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# Asymptotic solutions for the asymmetric flow in a channel with porous retractable walls under a transverse magnetic field*

Hongxia GUO[1], Ping LIN[1,2,†], Lin LI[3]

1. Beijing Key Laboratory for Magneto-Photoelectrical Composite and Interface Science, School of Mathematics and Physics, University of Science and Technology Beijing, Beijing 100083, China;
2. Division of Mathematics, University of Dundee, Dundee DD1 4HN, United Kingdom;
3. School of Mathematics and Physics, University of South China, Hengyang 421001, Hunan Province, China

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**Abstract** The self-similarity solutions of the Navier-Stokes equations are constructed for an incompressible laminar flow through a uniformly porous channel with retractable walls under a transverse magnetic field. The flow is driven by the expanding or contracting walls with different permeability. The velocities of the asymmetric flow at the upper and lower walls are different in not only the magnitude but also the direction. The asymptotic solutions are well constructed with the method of boundary layer correction in two cases with large Reynolds numbers, i.e., both walls of the channel are with suction, and one of the walls is with injection while the other one is with suction. For small Reynolds number cases, the double perturbation method is used to construct the asymptotic solution. All the asymptotic results are finally verified by numerical results.

**Key words** laminar flow, asymmetric flow, asymptotic solution, porous and retractable channel, magnetic field

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## 1 Introduction

The studies of laminar flow through porous channels with retractable walls have received extensive attention in the fields of fluid and mathematics due to their close connections with massive biological and engineering problems, e.g., blood flow in vessels, nutrition liquid transport in biological organisms, mass transfer among blood, air, and tissue, uniformly distributed irrigation, and natural transpiration. The electrically conducting viscous fluid in a channel

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† Corresponding author, E-mail: plin@maths.dundee.ac.uk

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with permeable walls can be used to simulate the biological problems. Taking blood flow in vessels as an example, when the blood is considered as an electrically conducting fluid, Higashi et al.[1] showed that the magnetic field has a significant effect on the vascular system based on the experimental investigation.

Berman[2] analyzed the two-dimensional steady laminar flow of a viscous incompressible fluid through a porous channel with uniform injection or suction by full using the symmetry to simplify the model for the first time, and reduced the Navier-Stokes equations to a fourth-order nonlinear ordinary differential equation with a parameter $R$ (see Section 2 for its definition) and four boundary conditions. Numerous studies about the laminar flow in a channel with permeable walls followed. Yuan[3] derived an asymptotic solution for the large injection case. Terrill[4] improved the solution by considering the inner layer. Sellars[5] and Terrill[6] investigated the large suction cases, and derived an asymptotic solution. Afterwards, in order to precisely simulate the blood flow, Uchida and Aoki[7] investigated the unsteady laminar flow driven by a single contraction or expansion of the channel walls. Goto and Uchida[8] studied the laminar flow in a semi-infinite porous pipe, the radius of which varied with time. Majdalani et al.[9] studied the laminar flow between slowly expanding or contracting walls, and obtained the similarity solution to describe the transport of biological fluids. Majdalani and Zhou[10] investigated the large injection and suction cases in a channel with retractable walls, and obtained the asymptotic solutions. For more details, one may refer to Asghar et al.[11], Xu et al.[12], and Dauenhauer and Majdalani[13].

Suryaprakasrao[14] investigated the laminar flow of an electrically-conductive viscous fluid in a porous channel under a transverse magnetic field for the first time, and obtained the asymptotic solution for the case with a small suction Reynolds number and small magnetic field number. Terrill and Shrestha[15–16] and Shrestha[17] made further extension of Suryaprakasrao's work, and obtained the asymptotic solutions for large suction and injection Reynolds numbers and all values of the Hartamnn number. Based on these works, investigators began to take account of the wall motion[18].

The asymmetric laminar flow caused by different wall permeability can be traced back to Proudman[19], who proposed the asymmetric flow for the first time. Terrill and Shrestha[20–21] and Shrestha and Terrill[22] extended Proudman's work, and obtained a series of asymptotic solutions with the method of matched asymptotic expansions for large injection, large suction, and mixed cases. Cox[23] and King and Cox[24] considered the problem of steady and unsteady flow in a channel with only one porous wall. Zhang et al.[25] studied the asymmetric flow analytically and numerically. However, these asymmetric work did not consider the cases with wall motion and a transverse magnetic filed in a channel.

In this paper, we will investigate the general asymmetric flow of an incompressible viscous fluid through a porous and retractable channel with a transverse magnetic field, and present the asymptotic solutions. The paper is arranged as follows. In Section 2, the formulation of the problem is presented by reducing the Navier-Stokes equations into a nonlinear ordinary differential equation via a similarity transformation. In Section 3, the effects of the magnetic field on the solution is examined, and the asymptotic solutions for different orders of the Reynolds number and the Hartman number are obtained. In Section 4, the asymptotic solutions for large Reynolds numbers in the case that one wall of the channel is with injection while the other wall is with suction are constructed. In Section 5, an asymptotic solution for the case of slowly contracting and weak permeability is presented. In Section 6, all the obtained solutions are verified by the numerical solutions. The summarization is given in Section 7 finally.

## 2 Mathematical formulation

The equations of continuity and momentum for the unsteady laminar flow of an incompressible viscous and electrically conducting fluid through a porous and retractable channel in a transverse magnetic field are

$$\nabla \cdot \boldsymbol{V} = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{V} + \frac{1}{\rho}\boldsymbol{J} \times \boldsymbol{B}, \tag{2}$$

where $\boldsymbol{J}$ and $\boldsymbol{B}$ are given by the Maxwell equations

$$\nabla \times \boldsymbol{H} = 4\pi\boldsymbol{J}, \tag{3}$$

$$\nabla \times \boldsymbol{E} = 0, \tag{4}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{5}$$

and the Ohm law

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}). \tag{6}$$

In the above equations, $\boldsymbol{B} = \mu_{\mathrm{m}}\boldsymbol{H}$. The symbol $\boldsymbol{J}$ represents the current density, $\boldsymbol{E}$ is the electric field, $\boldsymbol{H}$ is the magnetic field, $\nu$ is the viscosity of the fluid, $\sigma$ is the electrical conductivity, and $\mu_{\mathrm{m}}$ is the magnetic permeability. An elongated rectangular channel with a sufficiently large aspect ratio of width $L$ to height $h(t)$ and one closed end is considered. Despite the finite body length, it is reasonable to assume that the length of the channel is semi-infinite in order to neglect the effect of the opening at the end[7]. The permeability of both walls, which expand or contract uniformly at a time-dependent rate $\dot{h}(t)$, is different. It is assumed that the fluid velocity is $-v_1$ at the lower wall and $-v_2$ at the upper wall. Furthermore, a constant magnetic field with the strength $H_0$ is applied perpendicular to the walls. We take $\boldsymbol{H} = (0, H_0, 0)$ and $\boldsymbol{E} = (0, E_0, 0)$. It is assumed that there is no external electric field and the effects of the magnetic and electric fields produced by the motion of the electrically conducting fluid can be neglected[14–16]. Then, the magnetic term $\boldsymbol{J} \times \boldsymbol{B}$ of the body force in Eq. (2) can be reduced to

$$\boldsymbol{J} \times \boldsymbol{B} = -\sigma {B_0}^2\boldsymbol{V}, \tag{7}$$

where $B_0 = \mu_{\mathrm{m}}H_0$. In the two-dimensional problem, let $\tilde{x}$ and $\tilde{y}$ be the coordinates measured along and perpendicular to the flow direction, respectively, $u$ and $v$ be the velocity components, i.e., $\boldsymbol{V} = (u, v, 0)$, in the $\tilde{x}$- and $\tilde{y}$-directions, respectively.

The boundary conditions satisfied by the flow are

$$\begin{cases} u(\tilde{x}, -h) = 0, \quad v(\tilde{x}, -h) = -v_1 = -A_1\dot{h}, \\ u(\tilde{x}, h) = 0, \quad v(\tilde{x}, h) = -v_2 = -A_2\dot{h}, \end{cases} \tag{8}$$

where $A_1$ and $A_2$ are constant measures of the permeability of the lower and upper walls, respectively. We introduce the following stream function[10]:

$$\phi = \frac{\nu\tilde{x}}{h}F(y, t), \tag{9}$$

where $y = \tilde{y}/h$ is the dimensionless height. Then, the velocity components are given by

$$u = \frac{\partial\phi}{\partial\tilde{y}} = \frac{\nu\tilde{x}}{h^2}F_y, \quad v = -\frac{\partial\phi}{\partial\tilde{x}} = -\frac{\nu}{h}F, \tag{10}$$

so that the continuity equation (1) is naturally satisfied. Assume $|v_2| \geqslant |v_1|$. Without loss of generality, substitute Eqs. (10) and (7) into Eqs. (1) and (2). Then, we have

$$f''' + \alpha(yf'' + 2f') + R(ff'' - f'^2) - M^2 f' = K(R) \tag{11}$$

with the boundary conditions

$$f(-1)=1-\alpha_2,\quad f'(-1)=0,\quad f(1)=1,\quad f'(1)=0, \tag{12}$$

where $\alpha$ is the wall expansion ratio defined by $\alpha=\frac{h\dot{h}}{\nu}$ (positive for expansion and negative for contraction), $R=\frac{v_2 h}{\nu}$ is the Reynolds number (positive for injection and negative for suction), $M=\mu_{\rm m}H_0 h(\frac{\sigma}{\rho\nu})^{\frac{1}{2}}$ is the Hartman number, $\alpha_2=1-\frac{v_1}{v_2}$ is an asymmetric parameter, and $K$ is an integration constant. Similar derivations for transforming the Navier-Stokes equation into Eq. (11) can be referred to Majdalani et al.[9]. The fluid flow through the channel is symmetric about the center line of the channel for $\alpha_2=2$. In this paper, we will mainly focus on the asymmetric flow.

## 3 Asymptotic solutions for the large suction case

In this section, we will consider the case that both walls of the channel are with large suction and the suction velocities on the upper and lower walls are different. Asymptotic solutions will be constructed for $\alpha=O(1)$ and $M^2=O(1)$ as $R\to-\infty$ and for $\alpha=O(1)$ and $M^2=O(R)$ as $R\to-\infty$, respectively.

### 3.1 Asymptotic solutions for large suction Reynolds numbers

For the large suction case, assume $v_1>0>v_2$. Take $\varepsilon=-\frac{1}{R}>0$ as a perturbation parameter. Then, Eq. (11) can be rewritten as follows:

$$\varepsilon f'''+\varepsilon\alpha(yf''+2f')-(ff''-f'^2)-\varepsilon M^2 f'=k, \tag{13}$$

where $\varepsilon K(R)=k$, which is the equation to be solved for the suction case subject to Eq. (12). If the channel wall is with large suction, there exists a boundary layer near the upper wall when $M=0$ and $\alpha_2=2$[10]. Therefore, there may be a boundary layer near both walls, and the correction near both walls may be needed. We thus use the method of boundary layer correction[27] to construct the asymptotic solution. The solution can be expanded as follows:

$$f=f_0(y)+\varepsilon(f_1(y)+g_1(\tau)+h_1(\eta))+\varepsilon^2(f_2(y)+g_2(\tau)+h_2(\eta))+\cdots. \tag{14}$$

The integrating constant $k$ can be written as follows:

$$k=k_0+\varepsilon k_1+\varepsilon^2 k_2+\cdots. \tag{15}$$

In the above equations, $\tau=\frac{1-y}{\varepsilon}$ and $\eta=\frac{1+y}{\varepsilon}$ are the stretching transformations near $y=1$ and $y=-1$, respectively, and $g_i(\tau)$ and $h_i(\eta)$ $(i=1,2,3,\cdots)$ are boundary layer functions which will rapidly decay when $y$ is away from $y=1$ and $y=-1$, respectively.

Substituting Eqs. (14) and (15) into Eq. (13) and equating the equal powers of $\varepsilon$, we have

$$\varepsilon^0: f_0'^2-f_0f_0''=k_0, \tag{16}$$

$$\varepsilon^1: f_0f_1''-2f_0'f_1'+f_0''f_1=\alpha(yf_0''+2f_0')-M^2f_0'+f_0'''-k_1, \tag{17}$$

$$\varepsilon^2: f_0f_2''-2f_0'f_2'+f_0''f_2=\alpha(yf_1''+2f_1')-M^2f_1'+f_1'''-f_1f_1''+f_1'^2-k_2, \tag{18}$$

$$\vdots$$

$$\varepsilon^{-1}: \dddot{g}_1+f_0(1)\ddot{g}_1=0, \tag{19}$$

$$\varepsilon^0: \dddot{g}_2+f_0(1)\ddot{g}_2=(\alpha-f_1(1))\ddot{g}_1+f_0'(1)\tau\ddot{g}_1-g_1\ddot{g}_1-2f_0'(1)\dot{g}_1+\ddot{g}_1^2=0, \tag{20}$$

$$\vdots$$

$$\varepsilon^{-1}: \dddot{h}_1-f_0(-1)\ddot{h}_1=0, \tag{21}$$

$$\varepsilon^0 : \dddot{h}_2 - f_0(-1)\ddot{h}_2 = (f_1(-1)+\alpha)\ddot{h}_1 + f_0'(-1)\eta\ddot{h}_1 + h_1\ddot{h}_1 - 2f_0'(-1)\dot{h}_1 - \dot{h}_1^2 = 0, \tag{22}$$

$$\vdots$$

where $f'$, $\dot{g}$, and $\dot{h}$ denote the derivatives with respect to $y$, $\tau$, and $\eta$, respectively.

$$\begin{cases} f_i(y) = f_i(1-\varepsilon\tau) = f_i(1) - \varepsilon\tau f_i'(1) + \dfrac{1}{2}\varepsilon^2\tau^2 f_i''(1) + \cdots, \\ f_i(y) = f_i(\varepsilon\eta - 1) = f_i(-1) + \varepsilon\eta f_i'(-1) + \dfrac{1}{2}\varepsilon^2\eta^2 f_i''(-1) + \cdots. \end{cases}$$

$g_j(\tau)h_j(\eta)$ is exponentially small, which is considered to be approximately zero in the rest of the section. Substituting Eq. (14) into Eq. (12). Then, the boundary conditions to be satisfied by $f_i(y)$, $g_i(\tau)$, and $h_i(\eta)$ at $y=1$ or $\tau=0$ and $y=-1$ or $\eta=0$ are

$$f_0|_{y=1} = 1, \quad f_0|_{y=-1} = 1-\alpha_2 = a, \tag{23}$$

$$f_{i-1}'|_{y=1} - \dot{g}_i|_{\tau=0} = 0, \quad i=1,2,\cdots, \tag{24}$$

$$f_{i-1}'|_{y=-1} + \dot{h}_i|_{\eta=0} = 0, \quad i=1,2,\cdots, \tag{25}$$

$$f_i|_{y=1} + g_i|_{\tau=0} = 0, \quad f_i|_{y=-1} + h_i|_{\eta=0} = 0, \quad i=1,2,\cdots, \tag{26}$$

where $a = \frac{v_1}{v_2}$, i.e., $-1 < a < 0$. One solution of Eq. (16) with Eq. (23) is

$$f_0 = \frac{1-a}{2}y + \frac{1+a}{2}, \quad k_0 = \frac{(a-1)^2}{4}. \tag{27}$$

Thus, Eq. (24) becomes

$$\dot{g}_1|_{\tau=0} = f_0'|_{y=1} = \frac{1-a}{2}. \tag{28}$$

The boundary layer solution of Eq. (19) satisfying Eq. (28) is

$$g_1 = \frac{a-1}{2}\mathrm{e}^{-\tau}. \tag{29}$$

The boundary layer solution of Eq. (21) satisfying Eq. (25) is

$$h_1 = \frac{a-1}{2a}\mathrm{e}^{a\eta}. \tag{30}$$

Substituting Eq. (27) into Eq. (17), we have

$$\frac{1}{2}((a-1)y - a - 1)f_1'' + (1-a)f_1' = \frac{1}{2}(2\alpha - M^2)(a-1) - k_1. \tag{31}$$

The corresponding boundary conditions from Eq. (26) are

$$f_1|_{y=1} = -g_1|_{\tau=0} = \frac{1-a}{2}, \quad f_1|_{y=-1} = -h_1|_{\eta=0} = \frac{1-a}{2a}. \tag{32}$$

Hence, the solution of $f_1$ subject to Eq. (32) is

$$\begin{aligned} f_1 = \frac{-1}{16a(a^2+a+1)}&(((a-1)^2(a^2+2M^2a-4\alpha a-2a+1) - 4a(a-1)k_1)y^3 \\ &- 3((a^2-1)(a^2+2M^2a-4\alpha a-2a+1) - 4a(a+1)k_1)y^2 \\ &+ ((a-1)^2(3a^2-2M^2a+4\alpha a+6a+3) - 4a(a-1)k_1)y \\ &+ ((a^2-1)(7a^2+6M^2a-12\alpha a-2a+7) - 12a(a+1)k_1)), \end{aligned} \tag{33}$$

where $k_1$ is still unknown.

Substituting Eqs. (27) and (33) into Eq. (18), we can obtain the expression of $f_2$. We find that one term of $f_2$ is

$$\frac{1}{8(a-1)^3(a^2+a+1)a}(M^2-4\alpha)((a^2+2M^2a-4\alpha a-2a+1)(a-1)-4ak_1)\cdot((1-a)y+a+1)^3\lg((1-a)y+a+1). \tag{34}$$

It is a secular term, and thus must be zero. Therefore,

$$k_1=\frac{a-1}{4a}(a^2+2M^2a-4\alpha a-2a+1).$$

The expression of $f_1$ becomes

$$f_1=-\frac{(a-1)^2}{4a}y+\frac{1-a^2}{4a}. \tag{35}$$

Then, according to Eq. (24), the boundary condition for $g_2$ becomes

$$\dot{g}_2|_{\tau=0}=f_1'|_{y=1}=\frac{-(a-1)^2}{4a}. \tag{36}$$

Substituting Eq. (29) into Eq. (20), we have

$$g_2=\frac{1-a}{8a}((a^2-a)\tau^2+2(3a^2-2\alpha a-3a)\tau+6a^2-4\alpha a-8a+2)\mathrm{e}^{-\tau}. \tag{37}$$

Similarly, the solution of $h_2$ becomes

$$h_2=\frac{a-1}{8a^3}(-(a^3-a^2)\eta^2+2(2\alpha a+3a-3)a\eta+2(a^2-2\alpha a-4a+3))\mathrm{e}^{a\eta}. \tag{38}$$

Hence, the boundary conditions for $f_2$ from Eq. (26) are

$$f_2|_{y=1}=-g_2|_{\tau=0}=\frac{a-1}{4a}(3a^2-2\alpha a-4a+1), \tag{39}$$

$$f_2|_{y=-1}=-h_2|_{\eta=0}=\frac{1-a}{4a^3}(a^2-2\alpha a-4a). \tag{40}$$

Substituting Eqs. (27) and (35) into Eq. (18), we have

$$\frac{1}{2}((a-1)y-a-1)f_2''+(1-a)f_2'+\frac{(a-1)^2}{16a^2}(a^2+(4M^2-8\alpha-2)a+1)=k_2. \tag{41}$$

Then, we have

$$f_2=\frac{a-1}{8a^3}((3a^4-(4+2\alpha)a^3+2a^2-(4+2\alpha)a+3)y+(a^2-1)(3a^2-(4+2\alpha)a+3)), \tag{42}$$

and

$$k_2=-\frac{(a-1)^2}{16a^3}(6a^4-(9+4\alpha)a^3-(4M^2-8\alpha-6)a^2-(9+4\alpha)a+6).$$

The asymptotic solution for the large suction case of $y \in [-1, 1]$ is

$$\begin{aligned} f(y) &= \frac{1-a}{2}y + \frac{1+a}{2} + \varepsilon\Big( -\frac{(a-1)^2}{4a}y + \frac{1-a^2}{4a} + \frac{a-1}{2}\mathrm{e}^{-\tau} + \frac{a-1}{2a}\mathrm{e}^{a\eta}\Big) \\ &\quad + \varepsilon^2\Big(\frac{a-1}{8a^3}((3a^4 - (4+2\alpha)a^3 + 2a^2 - (4+2\alpha)a + 3)y \\ &\quad + (a^2-1)(3a^2 - (4+2\alpha)a + 3)) + \frac{1-a}{8a}((a^2-a)\tau^2 \\ &\quad + 2(3a^2 - 2\alpha a - 3a)\tau + 6a^2 - 4\alpha a - 8a + 2)\mathrm{e}^{-\tau} + \frac{a-1}{8a^3}(-(a^3-a^2)\eta^2 \\ &\quad + 2(2\alpha a + 3a - 3)a\eta + 2(a^2 - 2\alpha a - 4a + 3))\mathrm{e}^{a\eta}\Big) + O(\varepsilon^3), \end{aligned} \tag{43}$$

where

$$\tau = \frac{1-y}{\varepsilon}, \quad \eta = \frac{1+y}{\varepsilon}.$$

### 3.2 Asymptotic solution for large suction Reynolds number $R$ and large Hartmann number $M$

It is clear from Eq. (43) that the effect of the magnetic field on the channel flow can be negligible. Therefore, in this section, we will consider the case of large magnetic field, which may have a noticeable effect on the flow. When the suction Reynolds number $R$ and the Hartmann number $M$ are both large and we take the same order of effects on the flow through the channel, there may be a combined viscous suction and magnetic boundary layer at both of the channel walls. Therefore, we assume $r = -\frac{M^2}{R} > 0$ and $r \sim O(1)$, and choose $\varepsilon = -\frac{1}{R}$ as the perturbation parameter. The method used in Subsection 3.1 can be applied to this case. Equation (11) can be rewritten as follows:

$$\varepsilon f''' + \varepsilon\alpha(yf'' + 2f') - (ff'' - f'^2) + rf' = k, \tag{44}$$

where $\varepsilon K(R) = k$. A solution of Eq. (44) satisfying the corresponding boundary conditions in Eq. (12) can be obtained by using the similar procedure as in the large suction case. Therefore, $f_i$, $g_i$, and $h_i$ are expressed as follows:

$$f_0 = \frac{1-a}{2}y + \frac{1+a}{2}, \tag{45}$$

$$f_1 = -\frac{(a-1)^2}{4a}y + \frac{1-a^2}{4a}, \tag{46}$$

$$\begin{aligned} f_2 &= \frac{a-1}{8a^3}((3a^4 + (2r - 2\alpha - 4)a^3 + 2a^2 - (4+2\alpha)a - 2r + 3)y \\ &\quad + 3a^4 + (2r - 2\alpha - 4)a^3 + (4+2\alpha)a + 2r - 3), \end{aligned} \tag{47}$$

$$g_1 = \frac{a-1}{2}\mathrm{e}^{-\tau}, \tag{48}$$

$$h_1 = \frac{a-1}{2a}\mathrm{e}^{a\eta}, \tag{49}$$

$$\begin{aligned} g_2 &= \frac{a-1}{8a}(-(a^2-a)\tau^2 - 2(3a^2 + (2r - 2\alpha - 3)a)\tau \\ &\quad - 2(3a^2 + 2(r - \alpha - 2)a + 1))\mathrm{e}^{-\tau}, \end{aligned} \tag{50}$$

$$h_2 = \frac{a-1}{8a^3}(-(a^3-a^2)\eta^2 + 2((3+2\alpha)a^2 + (2r-3)a)\eta + 2(a^2-(4+2\alpha)a-2r+3))\mathrm{e}^{a\eta}. \tag{51}$$

Then, the asymptotic solution for this case is

$$\begin{aligned} f(y) = &\frac{1-a}{2}y + \frac{1+a}{2} + \varepsilon\Big(-\frac{(a-1)^2}{4a}y + \frac{1-a^2}{4a} + \frac{a-1}{2}\mathrm{e}^{-\tau} + \frac{a-1}{2a}\mathrm{e}^{a\eta}\Big) \\ &+ \varepsilon^2\Big(\frac{a-1}{8a^3}((3a^4 + (2r-2\alpha-4)a^3 + 2a^2 - (4+2\alpha)a - 2r+3)y \\ &+ 3a^4 + (2r-2\alpha-4)a^3 + (4+2\alpha)a + 2r - 3) \\ &+ \frac{a-1}{8a}(-(a^2-a)\tau^2 - 2(3a^2 + (2r-2\alpha-3)a)\tau - 2(3a^2 + 2(r-\alpha-2)a+1))\mathrm{e}^{-\tau} \\ &+ \frac{a-1}{8a^3}(-(a^3-a^2)\eta^2 + 2((3+2\alpha)a^2 + (2r-3)a)\eta + 2(a^2-(4+2\alpha)a-2r \\ &+ 3))\mathrm{e}^{a\eta}\Big) + O(\varepsilon^3), \end{aligned} \tag{52}$$

where

$$\tau = \frac{1-y}{\varepsilon}, \quad \eta = \frac{1+y}{\varepsilon}, \quad -1 < a < 0.$$

From Eq. (52), we know that the large magnetic field has a noticeable effect on the flow.

## 4 Asymptotic solutions for the mixed cases

For the asymmetric model, we will consider the case where injection and suction are mixed at the upper and lower walls. The mixed cases are either mixed injection or mixed suction for the porous channel flow considered by Terrill and Shrestha[21]. The flow governed by Eq. (11) is of mixed injection for positive $v_1$ and $v_2$, while is of mixed suction for negative $v_1$ and $v_2$. Both of the asymptotic expressions are presented in the subsequent subsections, respectively.

### 4.1 Asymptotic solution for the mixed injection case

For the mixed injection case, it is assumed that $v_2 \geqslant v_1 > 0$, i.e., $0 < a \leqslant 1$. Then, the equation, satisfying Eq. (12), can be rewritten as follows:

$$\varepsilon f''' + \varepsilon\alpha(yf'' + 2f') + (ff'' - f'^2) - \varepsilon M^2 f' = k, \tag{53}$$

where $\varepsilon = \frac{1}{R} > 0$ can be treated as a small parameter, and $k$ is an arbitrary constant.

The wall at $y = -1$ is with suction, which may produce a boundary layer, while the other wall at $y = 1$ is with injection. One correction term may have to be introduced due to the presence of the boundary layer at $y = -1$. Thus, $f(y)$ and $k$ are expanded as follows:

$$f(y) = f_0(y) + \varepsilon(f_1(y) + h_1(\eta)) + \varepsilon^2(f_2(y) + h_2(\eta)) + \cdots, \tag{54}$$

$$k = k_0 + \varepsilon k_1 + \varepsilon k_2 + \cdots, \tag{55}$$

where $\eta = \frac{1+y}{\varepsilon}$ is the stretching transformation near $y = -1$, and $h_i(\eta)$ $(i = 1, 2, \cdots)$ are boundary layer functions (rapidly decay when $y$ is away from $-1$). Substitute Eq. (54) into Eq. (12). Then, the boundary conditions become

$$f_0|_{y=1} = 1, \quad f_0'|_{y=1} = 0, \quad f_0|_{y=-1} = 1 - \alpha_2 = a, \tag{56}$$

$$f_{i-1}'|_{y=-1} + \dot{h}_i|_{\eta=0} = 0, \quad i = 1, 2, \cdots, \tag{57}$$

$$f_i|_{y=1} = 0, \quad f_i'|_{y=1} = 0, \quad f_i|_{y=-1} + h_i|_{\eta=0} = 0, \quad i = 1, 2, \cdots . \tag{58}$$

Substituting Eqs. (54) and (55) into Eq. (53), we have

$$f_0 f_0'' - f_0'^2 = k_0 \tag{59}$$

subject to Eq. (56) at $O(1)$.

It can be easily verified that the leading order solution is $f_0 = \cos(by - b)$, which is a periodic function. Then, we have

$$b = \frac{1}{2}(\arccos a + 2n\pi), \quad k_0 = -b^2 = -\frac{1}{4}(\arccos a + 2n\pi)^2, \quad n = 0, 1, 2 \cdots ,$$

from which we can obtain many solutions of Eq. (59). $f_0'$, which is proportional to the streamwise velocity $u$, does not change its sign when $n = 0$. However, $f_0'$ does change its sign at least three times when $n = 1$, i.e., the streamwise fluid flow direction will change at least three times. $f_0'$ does change its sign at least five times when $n \geqslant 2$. These phenomena may not occur physically. The laboratory experiment on a porous pipe without expansion or contraction has been conducted by Wageman and Guevara[26], and the zeroth-order asymptotic solution $f_0 = \frac{(-1)^n}{B} \sin \frac{(2n+1)\pi X}{2}$ $(n = 0, 1, 2, \cdots)$ for the injection case is obtained. The only solution that has been experimentally observed is in Ref. [26], where $n = 0$. Meanwhile, Proudman[19] pointed out that $f$ could have at most one zero, and it was common for any combination of the signs $v_1$ and $v_2$ for the reduced solution. However, there is at least two zeros when $n > 0$. Therefore, there must be $n = 0$.

Now, we consider $f_0 = \cos(by - b)$ as the leading order solution. When the terms of $O(\varepsilon^{-1})$ are collected, the equation for $h_1$ becomes

$$\dddot{h}_1 + f_0(-1)\ddot{h}_1 = 0 \tag{60}$$

and satisfies Eq. (57), i.e.,

$$\dot{h}_1|_{\eta=0} = -f_0'|_{y=-1} = -b\sin(2b). \tag{61}$$

Hence, the solution $h_1$ is

$$h_1 = \frac{b}{a}\sin(2b)\mathrm{e}^{-a\eta}. \tag{62}$$

When the terms of $O(\varepsilon)$ are collected, the differential equation for $f_1$ becomes

$$f_0 f_1'' - 2f_0' f_1' + f_0'' f_1 = -f_0''' - \alpha(y f_0'' + 2f_0') + M^2 f_0' + k_1. \tag{63}$$

The boundary conditions in Eq. (58) are

$$f_1|_{y=1} = 0, \quad f_1'|_{y=1} = 0, \quad f_1|_{y=-1} = -h_1|_{\eta=0} = -\frac{b}{a}\sin(2b). \tag{64}$$

To simplify the equation, let $z = by - b$. Then, Eq. (63) can be rewritten as follows:

$$b\cos z f_1'' + 2b\sin z f_1' - b\cos z f_1 = (b+z)\alpha\cos z + (2\alpha - M^2 - b^2)\sin z + \lambda, \tag{65}$$

where $'$ denotes the derivative with respect to $z$, and $\lambda = \frac{k_1}{b}$. The boundary conditions become

$$f_1|_{z=0} = 0, \quad f_1'|_{z=0} = 0, \quad f_1|_{z=-2b} = -\frac{b}{a}\sin(2b). \tag{66}$$

To construct the solution of Eq. (65), we start with the corresponding homogeneous equation as follows:

$$b\cos z f_{1h}'' + 2b\sin z f_{1h}' - b\cos z f_{1h} = 0. \tag{67}$$

We can easily obtain its solution as follows:

$$f_{1h} = P_1 \sin z + P_0 (z \sin z + \cos z), \tag{68}$$

where $P_0$ and $P_1$ are two arbitrary constants. Applying the method of variation of parameters, we can find the solution of Eq. (68) in terms of

$$f_1 = P_1(z) \sin z + P_0(z)(z \sin z + \cos z). \tag{69}$$

Through the standard process, we have

$$\begin{aligned} P_0 = & \frac{1}{2b}((b^2 + M^2 - 4\alpha)(\ln(1 - \sin z) - \ln \cos z) + (b^2 + M^2 - 2\alpha)\sec^2 z \sin z \\ & - \lambda - 2\alpha(b + z)\cos z + b^2) + n_0, \end{aligned} \tag{70}$$

$$P_1 = \frac{1}{2b}(-2(b^2 + M^2 - 2\alpha - b\alpha z)\sec z + \lambda z \sec^2 z + \lambda \tan z) + Q(z) + n_1, \tag{71}$$

where

$$Q(z) = \frac{\alpha}{b} z^2 \sec z + \frac{1}{b}\int_0^z ((b^2 + M^2 + \alpha)\phi \sec \phi - (b^2 + M^2 - 2\alpha)\phi \sec^3 \phi)\mathrm{d}\phi, \tag{72}$$

and $n_0$ and $n_1$ are constants to be determined.

Substituting Eqs. (70) and (71) into Eq. (69), we have

$$\begin{aligned} f_1 = & -\alpha - \frac{\alpha}{b} z + (Q(z) + n_1)\sin z + n_0 z \sin z + n_0 \cos z - \frac{b^2 + M^2 - 2\alpha}{2b}\tan z - \frac{\alpha}{b}\tan z \\ & + \frac{\lambda}{2b}\sin z \tan z - \frac{\lambda}{2b}\sec z + \frac{b^2 + M^2 - 2\alpha}{2b} z \tan^2 z + \frac{b^2 + M^2 - 4\alpha}{2b}(\ln(1 - \sin z) \\ & - \ln \cos z)(z \sin z + \cos z). \end{aligned} \tag{73}$$

According to Eq. (66) and taking account of $Q(0) = 0$, we have

$$n_0 = \alpha + \frac{\lambda}{2b}, \quad n_1 = \frac{b^2 + M^2 - 2\alpha}{b}, \tag{74}$$

$$\begin{aligned} \lambda = & \frac{1}{2a(b \sin(2b) + \cos(2b))}(2ab\alpha + (2b^2 - 2ab^2 - 2aM^2 + 4a\alpha - 4ab^2\alpha \\ & - 2abQ(-2b))\sin(2b) - 2ab\alpha \cos(2b) + (ab^2 + aM^2 - 2a\alpha + 8ab^2\alpha)\tan(2b) \\ & + (-2ab^3 - 2abM^2 + 4ab\alpha)\tan^2(2b) \\ & + a(b^2 + M^2 - 4\alpha)(\cos(2b) + 2b \sin(2b))(\ln(1 + \sin(2b)) - \ln \cos(2b))). \end{aligned} \tag{75}$$

Hence, the solution of Eq. (63) is

$$\begin{aligned} f_1 = & -\alpha - \frac{\alpha}{b} z + \Big(Q(z) + \frac{b^2 + M^2 - 2\alpha}{b}\Big)\sin z + \Big(\alpha + \frac{\lambda}{2b}\Big) z \sin z + \Big(\alpha + \frac{\lambda}{2b}\Big)\cos z \\ & - \frac{b^2 + M^2 - 2\alpha + 2\alpha z^2}{2b}\tan z + \frac{\lambda}{2b}\sin z \tan z - \frac{\lambda}{2b}\sec z + \frac{b^2 + M^2 - 2\alpha}{2b} z \tan^2 z \\ & + \frac{b^2 + M^2 - 4\alpha}{2b}(\ln(1 - \sin z) - \ln \cos z)(z \sin z + \cos z), \end{aligned} \tag{76}$$

where $\lambda$ is given by Eq. (75). An asymptotic solution for the mixed injection for $v_2 \geqslant v_1 > 0$, i.e., $0 < a \leqslant 1$, is

$$\begin{aligned} f(y) = {} & \cos z + \varepsilon\Big( -\alpha - \frac{\alpha}{b} z + \Big(Q(z) + \frac{b^2 + M^2 - 2\alpha}{b}\Big)\sin z + \Big(\alpha + \frac{\lambda}{2b}\Big) z \sin z \\ & + \Big(\alpha + \frac{\lambda}{2b}\Big)\cos z - \frac{b^2 + M^2 - 2\alpha + 2\alpha z^2}{2b}\tan z + \frac{\lambda}{2b}\sin z \tan z \\ & - \frac{\lambda}{2b}\sec z + \frac{b^2 + M^2 - 2\alpha}{2b} z \tan^2 z + \frac{b^2 + M^2 - 4\alpha}{2b}(\ln(1 - \sin z) - \ln\cos z) \\ & \cdot (z\sin z + \cos z) + \frac{b}{a}\sin(2b)\mathrm{e}^{-a\eta}\Big) + O(\varepsilon^2), \end{aligned} \tag{77}$$

where

$$\eta = \frac{1+y}{\varepsilon}, \quad z = by - b.$$

**Remark 1** We remark that $v_1 \geqslant v_2 > 0$, i.e., $a \geqslant 1$. We can have the solution of the format $f_0 = \cosh(dy - d)$, where

$$d = \pm\frac{1}{2}\operatorname{arccosh} a, \quad k_0 = d^2 = \frac{1}{4}(\operatorname{arccosh} a)^2.$$

Since $\cosh x$ is an even function, the solutions for $d = \frac{1}{2}\operatorname{arccosh} a$ and $d = -\frac{1}{2}\operatorname{arccosh} a$ are the same, and it is also means that there is only one solution of Eq. (59).

### 4.2 Asymptotic solution for the mixed suction case

For the mixed suction case, it is assumed that $v_2 \leqslant v_1 < 0$, i.e., $0 < a \leqslant 1$. The equation is the same as Eq. (13), which satisfies Eq. (12). Correction terms may have to be introduced due to a possible boundary layer near $y = 1$. Thus, $f(y)$ and $k$ are expanded as follows:

$$f(y) = f_0(y) + \varepsilon(f_1(y) + g_1(\tau)) + \varepsilon^2(f_2(y) + g_2(\tau)) + \cdots, \tag{78}$$

$$k = k_0 + \varepsilon k_1 + \varepsilon k_2 + \cdots, \tag{79}$$

where $\varepsilon = -\frac{1}{R} > 0$, $\tau = \frac{1-y}{\varepsilon}$ is the stretching transformation near $y = 1$, and $g_i(\tau)$ $(i = 1, 2, \cdots)$ are boundary layer functions (rapidly decay when $y$ is away from 1). The corresponding boundary conditions become

$$f_0|_{y=1} = 1, \quad f_0|_{y=-1} = 1 - \alpha_2 = a, \quad f_0'|_{y=-1} = 0, \tag{80}$$

$$f_{i-1}'|_{y=1} - \dot{g}_i|_{\tau=0} = 0, \quad i = 1, 2, \cdots, \tag{81}$$

$$f_i|_{y=-1} = 0, \quad f_i'|_{y=-1} = 0, f_i|_{y=1} + g_i|_{\tau=0} = 0, \quad i = 1, 2, \cdots. \tag{82}$$

Substituting Eqs. (78) and (79) into Eq. (13) and collecting the same power of $\varepsilon$, we have

$$\varepsilon^0 : f_0'^2 - f_0 f_0'' = k_0, \tag{83}$$

$$\varepsilon^{-1} : \dddot{g}_1 + \ddot{g}_1 = 0, \tag{84}$$

$$\varepsilon^1 : f_0 f_1'' - 2f_0' f_1' + f_0'' f_1 = \alpha(y f_0'' - 2f_0') - M^2 f_0' + f_0''' - k_1, \tag{85}$$

$$\vdots$$

Equations (83)–(85) subject to the boundary conditions in Eqs. (80)–(82), respectively. It can be easily obtained that the leading order solution is

$$f_0 = a\cosh(by + b), \tag{86}$$

where

$$b = \frac{1}{2}\operatorname{arccosh}\frac{1}{a}, \quad k_0 = -(ab)^2 = -\frac{1}{4}a^2\Big(\operatorname{arccosh}\frac{1}{a}\Big)^2.$$

Applying the similar method in the mixed injection case and solving Eqs. (84) and (85), we have

$$g_1 = -ab\sinh(2b)\mathrm{e}^{-\tau}, \tag{87}$$

$$\begin{aligned} f_1(z) = {} & -\alpha + \frac{\alpha}{b}z - \Big(S(z) + \frac{M^2 - b^2 - 2\alpha}{b}\Big)\sinh z - \frac{\mu}{2b}\cosh z \\ & - \frac{b^2 - M^2 + 2\alpha + 2\alpha z^2}{2b}\tanh z - \frac{b^2 - M^2 + 2\alpha}{2b}z\tanh^2 z \\ & - \frac{z\sinh z - \cosh z}{2b}(2b\alpha + \mu - 2(b^2 - M^2 + 4\alpha)\arctan(\tanh(z/2))), \end{aligned} \tag{88}$$

where

$$S(z) = \frac{1}{b}\int_0^z (-\alpha\phi\operatorname{sech}\phi + \alpha\phi^2\operatorname{sech}\phi\tanh\phi + (b^2 - M^2 + 2\alpha)\phi\operatorname{sech}\phi\tanh^2\phi)\mathrm{d}\phi, \tag{89}$$

$$\begin{aligned} \mu = {} & b(1 - a - 2\alpha) - S(2b) - \frac{M^2 - 2\alpha}{b} + \frac{M^2 - b^2 - 2\alpha - 8b^2\alpha}{2b\cosh(2b)} \\ & + \alpha\coth(2b) + (M^2 - b^2 - 2\alpha)\frac{\tanh(2b)}{\cosh(2b)} + \frac{\alpha}{2\sinh b\cosh(2b)} \\ & + \frac{M^2 - b^2 - 4\alpha}{b}\arctan(\tanh b)(\coth(2b) - 2b). \end{aligned} \tag{90}$$

An asymptotic solution of mixed suction for $v_2 \leqslant v_1 < 0$, i.e., $0 < a \leqslant 1$, is

$$\begin{aligned} f(y) = {} & \cosh z + \varepsilon\Big( - \alpha + \frac{\alpha}{b}z - \Big(S(z) + \frac{M^2 - b^2 - 2\alpha}{b}\Big)\sinh z - \frac{\mu}{2b}\cosh z \\ & - \frac{b^2 - M^2 + 2\alpha + 2\alpha z^2}{2b}\tanh z - \frac{b^2 - M^2 + 2\alpha}{2b}z\tanh^2 z \\ & - \frac{z\sinh z - \cosh z}{2b}(2b\alpha + \mu - 2(b^2 - M^2 + 4\alpha)\arctan\tanh(z/2)) \\ & - ab\mathrm{e}^{-\tau}\sinh(2b)\Big) + O(\varepsilon^2), \end{aligned} \tag{91}$$

where $\tau = \frac{1-y}{\varepsilon}$, and $z = by + b$.

**Remark 2** we remark that $v_1 \leqslant v_2 < 0$, i.e., $a \geqslant 1$, $a \geqslant 1$, and $f_0 = a\cos(dy + d)$ is a leading order solution, where

$$b = \frac{1}{2}\Big( \arccos\frac{1}{a} + 2n\pi\Big), \quad k_0 = (ab)^2 = \frac{1}{4}a^2\Big( \arccos\frac{1}{a} + 2n\pi\Big)^2.$$

## 5 Asymptotic solution for the case of small Reynolds $R$ and small expansion ratio $\alpha$

For the small injection or suction and small expansion ratio case, it is reasonable to use the method of regular perturbation expansion to construct the asymptotic solution. One, thus, may treat $R$ as a perturbation parameter. The solution may be expanded as follows:

$$f = f_0 + R f_1 + R^2 f_2 + \cdots . \tag{92}$$

Differentiating Eq. (11) with respect to $y$, we have

$$f'''' + \alpha(y f''' + 3f'') + R(f''' f - f'' f') - M^2 f'' = 0, \tag{93}$$

which satisfies Eq. (12). Substituting Eq. (92) into Eq. (93) and equating like powers of $R$, we have

$$f_0'''' + \alpha(y f_0''' + 3f_0'') - M^2 f_0'' = 0, \tag{94}$$

$$f_1'''' + \alpha(y f_1''' + 3f_1'') - M^2 f_1'' + f_0''' f_0 - f_0'' f_0' = 0. \tag{95}$$

The corresponding boundary conditions become

$$f_0|_{y=1} = 1, \quad f_0'|_{y=1} = 0, \quad f_0|_{y=-1} = 1 - \alpha_2, \quad f_0'|_{y=-1} = 0, \tag{96}$$

$$f_1|_{y=1} = 0, \quad f_1'|_{y=1} = 0, \quad f_1|_{y=-1} = 0, \quad f_1'|_{y=-1} = 0. \tag{97}$$

We consider the case where $\alpha$ is also small, and take it as the secondary perturbation parameter. Then, $f_0$ and $f_1$ can be further expanded as follows:

$$f_0 = f_{00} + \alpha f_{01} + O(\alpha^2), \tag{98}$$

$$f_1 = f_{10} + \alpha f_{11} + O(\alpha^2). \tag{99}$$

Substituting Eq. (98) into Eq. (94) and collecting like powers of $\alpha$, we have

$$f_{00}'''' - M^2 f_{00}'' = 0, \tag{100}$$

$$f_{01}'''' - M^2 f_{01}'' + y f_{00}''' + 3 f_{00}'' = 0 \tag{101}$$

with the boundary conditions

$$f_{00}|_{y=1} = 1, \quad f_{00}'|_{y=1} = 0, \quad f_{00}|_{y=-1} = 1 - \alpha_2, \quad f_{00}'|_{y=-1} = 0, \tag{102}$$

$$f_{01}|_{y=1} = 0, \quad f_{01}'|_{y=1} = 0, \quad f_{01}|_{y=-1} = 0, \quad f_{01}'|_{y=-1} = 0. \tag{103}$$

Thus, we have

$$f_{00} = \frac{\alpha_2(\sinh(My) - yM\cosh M)}{2(\sinh M - M\cosh M)} + \frac{2-\alpha_2}{2}, \tag{104}$$

$$\begin{aligned} f_{01} = \frac{D}{8M}(&y^2(M\cosh M - \sinh M)M\sinh(My) - y((M\cosh M - \sinh M)\cosh(My) \\ &+ \cosh M \sinh M + M\cosh(2M) - 2M) - (M\cosh M + 2\sinh M)M\sinh(My)), \end{aligned} \tag{105}$$

where $D = \frac{\alpha_2}{(M\cosh M - \sinh M)^2}$.

Substituting Eqs. (99) and (98) into Eq. (95) and equating equal powers of $\alpha$, we have

$$f_{10}'''' - M^2 f_{10}'' + f_{00}''' f_{00} - f_{00}'' f_{00}' = 0 \tag{106}$$

with the boundary condition

$$f_{10}|_{y=1} = 0, \quad f_{10}'|_{y=1} = 0, \quad f_{10}|_{y=-1} = 0, \quad f_{10}'|_{y=-1} = 0. \tag{107}$$

The solution of $f_{10}$ is

$$\begin{aligned} f_{10} = {} & Gy^2(2\alpha_2 M^2 \sinh(2M)\sinh(My)(M\cosh M - \sinh M)) \\ & + MGy(8(\alpha_2 - 2)(M\cosh M - \sinh M)^2 \sinh(My) \\ & + 28\alpha_2 \cosh M(M\cosh M - \sinh M)\cosh(My) + \alpha_2(14\sinh(2M)\cosh M \\ & - 2M\cosh(3M) - 26M\cosh M)) + 2G(4(\alpha_2 - 2)\sinh^2 M(M\cosh^2 M \\ & - \cosh M(M\cosh(My) + \sinh M) + \sinh M(\cosh(My) + M\sinh M)) \\ & - M^2 \sinh(2M)((\alpha_2 - 2)(2\cosh M\cosh(My) + \cosh(2M)) \\ & - 3\alpha_2(2\sinh M\sinh(My) + 1) + 6) \\ & + 2M^3\cosh^2 M(2(\alpha_2 - 2)(\cosh M\cosh(My) - 1) \\ & - \alpha_2 \sinh M\sinh(My))), \end{aligned} \tag{108}$$

where

$$G = \frac{\alpha_2}{64M\sinh M(M\cosh M - \sinh M)^3}.$$

Finally, the solution becomes

$$\begin{aligned} f(y) = {} & \frac{\alpha_2}{2p}(\sinh(My) - yM\cosh M) + \frac{2-\alpha_2}{2} + \frac{\alpha\alpha_2}{8Mp^2}(y^2 p - \sinh M)M\sinh(My) \\ & - y(p\cosh(My) + \cosh M\sinh M + M\cosh(2M) - 2M) \\ & - (M\cosh M + 2\sinh M)M\sinh(My)) \\ & + \frac{R\alpha_2}{64p^3 M\sinh M}(y^2(2\alpha_2 M^2\sinh(2M)\sinh(My)p) \\ & + My(8(\alpha_2 - 2)p^2\sinh(My) + 28\alpha_2 p\cosh M\cosh(My) \\ & + \alpha_2(14\sinh(2M)\cosh M - 2M\cosh(3M) - 26M\cosh M)) \\ & + 2(4(\alpha_2 - 2)\sinh^2 M(M\cosh^2 M - \cosh M(M\cosh(My) + \sinh M) \\ & + \sinh M(\cosh(My) + M\sinh M)) - M^2\sinh(2M)((\alpha_2 - 2)(2\cosh M\cosh(My) \\ & + \cosh(2M)) - 3\alpha_2(2\sinh M\sinh(My) + 1) + 6) \\ & + 2M^3\cosh^2 M(2(\alpha_2 - 2)(\cosh M\cosh(My) - 1) - \alpha_2\sinh M\sinh(My)))), \end{aligned} \tag{109}$$

where

$$p = M\cosh M - \sinh M.$$

## 6 Comparison of the asymptotic and numerical solutions

Terrill and Shrestha[20] pointed out that the comparison of $f''(1)$ was the most effective way. Hence, in this section, we will compare the asymptotic and numerical results of $f''(-1)$ and $f''(1)$ proportional to the skin-friction at the walls. The numerical solutions for Eqs. (11) and (12) can be obtained by a MATLAB boundary layer problem solver BVP4C.

From Table 1, we can see that the analytical results agree well with the numerical results for arbitrary constants $\alpha_2$ and $\alpha$ for $M^2/R \sim O(\varepsilon)$ in the large suction case. Nevertheless, it is clear from the last four lines of Table 1 that the asymptotic results deteriorate when $M^2$ becomes of the same order as that of the suction Reynolds number. Hence, we need to use the asymptotic solution for the case of $M^2/R \sim O(1)$. We find that the large magnetic field has a noticeable effect on the solution because of the parameter $r$ existing in $\varepsilon^2$. As shown in Table 2, when the solution (52) is used, the accuracy is largely improved compared with Table 2 for large $M$.

**Table 1** Comparison of $f''(-1)$ and $f''(1)$ for large suction $R$

| $M$ | $\alpha_2$ | $\alpha$ | $R$ | $f''(-1)$ | | $f''(1)$ | |
|---|---|---|---|---|---|---|---|
| | | | | Eq. (43) | Numerical | Eq. (43) | Numerical |
| 1.0 | 1.87 | −1.5 | −50 | 40.939 5 | 40.866 3 | −47.408 9 | −47.406 0 |
| 1.0 | 1.95 | −2.0 | −67 | 62.958 1 | 62.952 6 | −66.374 4 | −66.382 5 |
| 1.1 | 1.73 | −3.0 | −88 | 56.860 9 | 56.874 6 | −78.243 5 | −78.299 7 |
| 3.5 | 1.81 | 2.0 | −90 | 62.961 2 | 62.906 7 | −79.013 1 | −79.039 8 |
| 6.5 | 1.50 | 3.5 | −99 | 32.812 5 | 32.987 1 | −71.625 0 | −71.958 7 |
| 3.8 | 1.77 | 4.0 | −110 | 70.168 4 | 70.058 3 | −93.260 7 | −93.264 1 |
| 2.5 | 1.60 | −8.0 | −135 | 69.706 7 | 69.933 6 | −114.187 0 | −114.316 1 |
| 6.0 | 1.50 | 3.0 | −150 | 52.312 5 | 52.382 6 | −110.250 0 | −110.442 6 |
| 20.0 | 1.70 | 5.0 | −150 | 83.658 2 | 86.894 3 | −122.837 1 | −125.110 2 |
| 30.0 | 1.80 | −7.0 | −150 | 113.085 0 | 119.495 9 | −140.693 0 | −145.880 2 |
| 20.0 | 1.70 | 5.0 | −175 | 98.533 2 | 101.291 9 | −144.087 1 | −146.027 7 |
| 30.0 | 1.80 | −7.0 | −175 | 131.085 0 | 136.665 9 | −163.193 0 | −167.693 5 |

**Table 2** Comparison of $f''(-1)$ and $f''(1)$ for large suction $R$ and $M$

| $M$ | $\alpha_2$ | $\alpha$ | $R$ | $f''(-1)$ | | $f''(1)$ | |
|---|---|---|---|---|---|---|---|
| | | | | Eq. (52) | Numerical | Eq. (52) | Numerical |
| 20 | 1.7 | 5 | −150 | 86.896 3 | 86.894 3 | −125.103 8 | −125.110 2 |
| 20 | 1.7 | 5 | −175 | 101.308 7 | 101.291 9 | −146.030 0 | −146.027 7 |
| 30 | 1.8 | −7 | −150 | 119.835 0 | 119.495 9 | −146.092 5 | −145.880 2 |
| 30 | 1.8 | −7 | −175 | 136.870 7 | 136.665 9 | −167.821 1 | −167.693 5 |
| 20 | 1.7 | 5 | −213 | 123.423 6 | 123.390 4 | −177.983 4 | −177.975 0 |
| 30 | 1.8 | −7 | −213 | 188.936 5 | 188.869 1 | −208.964 9 | −208.911 4 |

From Tables 3 and 4, we know that the asymptotic solutions agree well with the numerical solutions in the mixed cases. Meanwhile, the skin-friction increases when $R$ increases.

At this juncture, it is useful to note that if there is no magnetic field, i.e., $M = 0$, and the physic model is symmetric, i.e., $\alpha_2 = 2$, the solution of small $R$ and $\alpha$ can be reduced to the solution given by Majdalani et al.[9]. Furthermore, it is clear from Table 5 that the asymptotic results have a good match to the numerical results, and the accuracy increases when $R$ and $\alpha$ decrease.

**Table 3** Comparisons of $f''(-1)$ and $f''(1)$ for mixed injection

| $M$ | $\alpha_2$ | $\alpha$ | $R$ | $f''(-1)$ | | $f''(1)$ | |
|---|---|---|---|---|---|---|---|
| | | | | Eq. (77) | Numerical | Eq. (77) | Numerical |
| 2 | 0.2 | −3 | 50 | 7.669 3 | 7.835 2 | −0.124 4 | −0.125 6 |
| 2 | 0.2 | −3 | 70 | 10.749 5 | 10.915 7 | −0.118 4 | −0.119 0 |
| 2 | 0.2 | −3 | 90 | 13.833 5 | 14.000 0 | −0.115 1 | −0.115 5 |
| 2 | 0.2 | −3 | 110 | 16.919 3 | 17.086 0 | −0.112 9 | −0.113 2 |
| 2 | 0.2 | −3 | 130 | 20.006 0 | 20.172 9 | −0.111 5 | −0.111 7 |
| 2 | 0.2 | −3 | 150 | 23.093 3 | 23.260 3 | −0.110 4 | −0.110 6 |
| 3 | 0.1 | 2 | 50 | 4.380 8 | 4.228 1 | −0.054 1 | −0.054 1 |
| 3 | 0.1 | 2 | 70 | 6.149 3 | 5.991 2 | −0.053 1 | −0.053 1 |
| 3 | 0.1 | 2 | 90 | 7.918 1 | 7.757 0 | −0.052 6 | −0.052 6 |
| 3 | 0.1 | 2 | 110 | 9.687 2 | 9.524 2 | −0.052 3 | −0.052 3 |
| 3 | 0.1 | 2 | 130 | 11.456 4 | 11.292 0 | −0.052 1 | −0.052 0 |
| 3 | 0.1 | 2 | 150 | 13.225 6 | 13.060 2 | −0.051 9 | −0.051 9 |

**Table 4** Comparisons of $f''(-1)$ and $f''(1)$ for mixed suction

| $M$ | $\alpha_2$ | $\alpha$ | $R$ | $f''(-1)$ | | $f''(1)$ | |
|---|---|---|---|---|---|---|---|
| | | | | Eq. (91) | Numerical | Eq. (91) | Numerical |
| 1 | 0.06 | −3.7 | −15 | 0.048 5 | 0.052 7 | −0.910 7 | −0.934 2 |
| 2 | 0.06 | −3.7 | −15 | 0.052 7 | 0.058 3 | −0.919 4 | −0.901 8 |
| 1 | 0.13 | 1.2 | −19 | 0.059 9 | 0.060 6 | −2.427 5 | −2.533 2 |
| 2 | 0.13 | 1.2 | −28 | 0.066 2 | 0.066 5 | −3.653 2 | −3.600 1 |
| 2 | 0.24 | 2.8 | −39 | 0.105 6 | 0.106 5 | −9.641 0 | −9.593 8 |
| 2 | 0.20 | −1.5 | −50 | 0.094 9 | 0.095 5 | −8.702 1 | −8.534 8 |
| 1 | 0.36 | 0.8 | −55 | 0.163 2 | 0.163 4 | −21.197 4 | −21.368 7 |
| 2 | 0.54 | 1.7 | −67 | 0.225 5 | 0.225 9 | −41.463 1 | −41.112 9 |
| 2 | 0.08 | −2.1 | −75 | 0.043 9 | 0.044 0 | −6.050 2 | −5.998 4 |
| 1 | 0.49 | 0.9 | −89 | 0.209 5 | 0.209 7 | −49.086 4 | −49.381 1 |

**Table 5** Comparisons of $f''(-1)$ and $f''(1)$ for small $R$ and $\alpha$

| $M$ | $\alpha_2$ | $\alpha$ | $R$ | $f''(-1)$ | | $f''(1)$ | |
|---|---|---|---|---|---|---|---|
| | | | | Eq. (109) | Numerical | Eq. (109) | Numerical |
| 3 | 1.9 | 0.01 | 0.01 | 4.234 9 | 4.234 9 | −4.234 2 | −4.234 2 |
| 3 | 1.8 | 0.02 | 0.03 | 4.003 5 | 4.003 5 | −3.999 6 | −3.999 6 |
| 5 | 1.5 | 0.04 | 0.06 | 4.663 6 | 4.663 6 | −4.649 5 | −4.649 7 |
| 7 | 1.3 | 0.10 | 0.09 | 5.266 5 | 5.266 7 | −5.242 6 | −5.243 0 |
| 9 | 1.0 | 0.15 | 0.19 | 5.026 2 | 5.026 3 | −4.972 7 | −4.973 5 |
| 12 | 0.8 | 0.22 | 0.28 | 5.204 2 | 5.204 3 | −5.130 9 | −5.131 9 |
| 15 | 0.6 | −0.35 | −0.30 | 4.857 5 | 4.857 6 | −4.925 0 | −4.926 0 |
| 20 | 0.4 | −0.40 | 0.29 | 4.272 6 | 4.273 0 | −4.223 8 | −4.223 8 |
| 13 | 0.3 | 0.50 | −0.38 | 2.048 2 | 2.049 0 | −2.100 7 | −2.100 7 |
| 8 | 0.1 | −0.30 | 0.40 | 0.476 5 | 0.476 8 | −0.454 8 | −0.454 8 |
| 8 | 0.1 | 0.50 | 0.60 | 0.458 0 | 0.458 1 | −0.425 5 | −0.425 4 |

## 7 Conclusions

In this paper, the similarity solutions of the asymmetric channel flow with porous retractable walls and a transverse magnetic field are constructed. Firstly, one asymptotic solution with the linear leading order is obtained for the large suction case. It is found that the skin-friction near the walls increases when the magnetic field intensity increases. Secondly, the asymptotic solutions are obtained for the flow in a channel, where one wall is with injection and the other wall is with suction. All the above asymptotic solutions are constructed for the most difficult large Reynolds number cases. Finally, one asymptotic solution is obtained by the two-parameter perturbation method when the wall contraction or expansion is weak and the injection or suction is small. All asymptotic solutions are verified by the numerical solutions obtained by the MATLAB boundary value problem solver BVP4C.

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