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A numerical study for multiple solutions of a singular boundary value problem arising from laminar flow in a porous pipe with moving wall

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ABSTRACT

This paper is concerned with multiple solutions of a singular nonlinear boundary value problem (BVP) on the interval [0, 1], which arises in a study of the laminar flow in a porous pipe with an expanding or contracting wall. For the singular nonlinear BVP, the correct boundary conditions are derived to guarantee that its linearization has a unique smooth solution. Then a numerical technique is proposed to find all possible multiple solutions. For the suction driven pipe flow with the expanding wall (e.g. $\alpha = 2$), we find a new solution numerically and classify it as a type VI solution. The computed results agree well with what can be obtained by the bifurcation package AUTO. In addition, we also construct asymptotic solutions for a few cases of parameters, which agree well with numerical solutions. These serve as validations of our numerical results. Thus we believe that the numerical technique designed in the paper is reliable, and may be further applied to solve a variety of nonlinear equations that arise from other flow problems.

1. Introduction

The laminar flow in a porous circular pipe or channel with an expanding or contracting wall has received considerable attention in recent years due to their relevance to a number of biological and engineering models, such as the transport of biological fluids through contracting or expanding vessels and the air circulation in the respiratory system. The earliest workers on the unsteady flow across an expanding wall can probably be traced back to Uchida and Aoki [1], in which the flow equations in a pipe are reduced to a single fourth-order nonlinear ordinary differential equation with the wall expansion ratio as a parameter. In order to simulate the laminar flow field in cylindrical solid rocket motors, Goto and Uchida [2] analyzed the laminar incompressible flow in a semi-infinite porous pipe whose radius varies with time. Following the route of this investigation, a variety of methods have been used to study this problem. For example, Boutros et al. [3,4] applied a Lie-group method to the equations of motion to determine symmetry reductions of partial differential equations. The resulting fourth-order nonlinear differential equation is then solved using small-parameter perturbations, and the results are compared with numerical solutions using shooting method. Asghar et al. [5] and Dinarvand and Rashidi [6] also discussed the flow in a slowly deforming channel with weak permeability using homotopy analysis method (HAM) and Adomian decomposition method (AMD), respectively.

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Besides the above mentioned results, multiple solutions are also found for governing equations in the porous pipe or channel with stationary walls. For example, Robinson [7] considered the inclusion of exponentially small terms in an asymptotic series to find two of the solutions analytically for the flow in a porous channel. Using the HAM, Xu et al. [8] recently investigated the multiple solutions of the flow in a porous channel with expanding or contracting walls and explored some ranges of the control parameters. However, little work is found in literature on multiple solutions of the laminar flow in a pipe with an expanding or contracting wall in a full range of Reynolds numbers. The main purpose of this paper is to find multiple solutions corresponding to governing equations. Since the transformation of governing equations to a singular nonlinear boundary value problem (BVP) can be found in previous literature (e.g. [9]), for the sake of simplicity, we only present the resulting BVP of the form

$$\eta f''' + f'' + \frac{\alpha}{2}(\eta f'' + f') + \frac{Re}{2}(f''f - f'^2) = k, \tag{1}$$

where $'$ denotes the derivative with respect to η, α and Re are wall expansion ratio and cross-flow Reynolds number, respectively, and k is an integration constant. Another form of Eq. (1) is often used to study its solution. It is obtained through a simple differentiation:

$$\eta f'''' + 2f''' + \frac{\alpha}{2}(\eta f''' + 2f'') + \frac{Re}{2}(f'''f - f''f') = 0. \tag{2}$$

The corresponding boundary conditions become

$$f'(1) = 0, \qquad f(1) = 1, \qquad f(0) = 0, \qquad \lim_{\eta\to 0} \eta^{\frac{1}{2}} f''(\eta) = 0. \tag{3}$$

Noting that when the wall is stationary (i.e. $\alpha = 0$), Terrill and Thomas [10] have presented the multiple solutions using a numerical technique, in which the boundary value problem is rewritten as an initial value problem at the left endpoint, and the initial values are updated to meet the boundary conditions at the other endpoint. This process is similar to that of shooting method. To overcome the singularity at $\eta = 0$, a Taylor expansion is used in the neighborhood of $\eta = 0$ and a Runge–Kutta method is then applied thereafter. Using the technique, Shankararaman and Liu [11] also considered the effect of the slip on existence and uniqueness of similarity solutions in a porous pipe with stationary walls. However for the laminar flow in a porous pipe with an expanding or contracting wall (i.e. $\alpha \neq 0$), if we continue to use their technique, the BVP (i.e. (2) and (3)) may not be easy to solve due to the singularity at $\eta = 0$ and multiple parameters (i.e. α and Re). Furthermore, we aim to find all possible multiple solutions for the full range of parameters. This largely increases the difficulty of the computation. So, to obtain multiple solutions of (2) and (3), it is necessary to design a new numerical technique.

In the current paper, we mainly focus on solving the problem (2) and (3) for multiple solutions. Since Eq. (2) has the singularity at $\eta = 0$, a solution to its linearization may blow up (see Section 2 for more details). Therefore, before solving it, we first analyze the singularity. There also exist plenty of papers dealing with the smoothness properties of solutions for singular BVPs. The problem (2) and (3) is a typical BVP with a singularity of the first kind [12]. Such BVPs often arise in numerous applications in natural sciences and engineering, e.g. when a partial differential equation (PDE) is reduced to an ordinary differential equation (ODE) by the cylindrical or spherical symmetry. Since the singular BVP is not evaluated easily at the singular point, the studies on it have become a recurring topic in the field of numerical calculation (e.g. [12–18]), where their attention mainly focus on the existence, uniqueness and smoothness of solutions. In particular, the structure of the boundary conditions which are necessary and sufficient for the linearization of a singular nonlinear BVP to have a reasonable smoothness of the solution on a closed interval including the singular point is of special interest. Other studies on the convergence properties of several finite difference, collocation and Galerkin schemes for singular BVPs may also be found in e.g. [12,16,19–22] and in a nice bibliography on solving singular BVP numerically [23].

The rest of the paper is organized as follows. In Section 2, we will focus first on setting appropriate boundary conditions such that the linearization of the problem (2) and (3) can have a reasonably smooth solution. To obtain all possible multiple solutions, we propose a technique in Section 3.1, where the problem (2) and (3) is converted into an initial value problem (IVP). The resulting IVP also has the singularity at $\eta = 0$. Therefore, in Section 3.2, according to a few results given in [24–28], we analyze the smoothness of the solution of the singular IVP near the singular point, and thus ODE solvers given in MATLAB can be used to solve it. Numerical results and multiple solutions are presented in Section 4. In order to further verify the numerical results, asymptotic solutions for some ranges of parameters are constructed by a few suitable perturbation methods and asymptotic results are compared with the numerical ones in Section 5. Finally, Section 6 concludes the paper.

2. The BVP with a singularity of the first kind

The problem we consider is singular and formulated as a two-point BVP, where correct boundary conditions can result in a well-posed BVP whose linearization has a unique smooth solution. This property is crucial when solving a nonlinear BVP numerically using the Newton method. So the following question arises: What boundary conditions may be derived to guarantee that the linearization of the problem (i.e. (2) and (3)) is to have a unique smooth solution in the interval [0, 1]?

Before we do such an analysis, the notation $C_q^p[0, 1]$ is introduced. It is the space whose elements have the form

$$\mathbf{x}(t) = (\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_q(t))^T, \quad 0 \le t \le 1, \tag{4}$$

where $\mathbf{x}_i(t),\ i = 1, 2, \ldots, q$, are p times continuously differentiable functions on [0, 1], and $C_q^p(0, 1]$ is defined in a similar way. For simplicity, we will remove the subscript q in the subsequent analysis and simply denote $C[0, 1] = C^0[0, 1],\ C(0, 1] = C^0(0, 1]$.

In general, for a BVP with a singularity of the first kind, its standard form is formulated as:

$$\mathbf{y}' = S\mathbf{y}/\eta + \mathbf{g}(\eta, \mathbf{y}), \quad 0 < \eta \le 1,\ \mathbf{y} \in C[0, 1] \cap C^1(0, 1], \tag{5}$$

$$\mathbf{b}(\mathbf{y}(0), \mathbf{y}(1)) = 0. \tag{6}$$

Here, $\mathbf{y}$ and $\mathbf{g}$ are vector-valued functions of dimension n, $\mathbf{b}$ is a vector-valued function of dimension $m \le n$. S is a constant $n \times n$ matrix. To write (2) and (3) into the standard form (5) and (6), we introduce the following new variables:

$$y_1 = \frac{f}{\eta}, \qquad y_2 = f', \qquad y_3 = f'', \qquad y_4 = \eta f''', \tag{7}$$

and collect them as a vector variable $\mathbf{y} = (y_1\ y_2\ y_3\ y_4)^T$. Note that $f(0) = 0$, thus y_1 is well defined. Then we can write Eq. (2) as Eq. (5), where

$$S = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{8}$$

and

$$\mathbf{g}(\eta, \mathbf{y}) = \left(0\ y_3\ 0\ -\frac{\alpha}{2}(y_4 + 2y_3) - \frac{Re}{2}(y_1y_4 - y_2y_3)\right)^T. \tag{9}$$

The problem (5) and (6) has been investigated by de Hoog and Weiss [16]. They have not only developed a canonical form for such BVPs, but also established a Fredholm theory for linear problems in this canonical form. Further, its analysis on more general problems can be found in the literature [12]. We will follow their technique in the subsequent analysis. It is assumed that the nonlinear two-point BVP (i.e. (2) and (3)) has an isolated solution $\mathbf{y}(\eta)$. This means that the linearized problem

$$\phi'(\eta) = S\phi(\eta)/\eta + \mathbf{A}(\eta)\phi(\eta), \quad 0 < \eta \le 1, \tag{10}$$

$$\mathbf{B}_0\phi(0) + \mathbf{B}_1\phi(1) = 0, \tag{11}$$

where

$$\mathbf{A}(\eta) = \frac{\partial \mathbf{g}(\eta, \mathbf{y}(\eta))}{\partial \mathbf{y}}; \qquad \mathbf{B}_i = \frac{\partial \mathbf{b}(\mathbf{y}(0), \mathbf{y}(1))}{\partial \mathbf{y}(i)}, \quad i = 0, 1; \tag{12}$$

has only the trivial solution. We note that $\frac{\partial \mathbf{g}}{\partial \mathbf{y}}$ is smooth, which indicates that the eigenvalues of S play a major role in posing the correct boundary conditions [16]. Before deriving the correct boundary conditions, we introduce some notations that will be used in the subsequent analysis.

Let X_0 and X_+ be the eigenspaces of S corresponding to the eigenvalue zero and the eigenvalues with positive real part, respectively, R and M be the projection matrices onto X_0 and X_+, respectively, and define

$$Q = I - R - M, \tag{13}$$

where I is an identity matrix. According to Lemmas 3.6, 3.7 and Theorem 3.1 in [16], the correct boundary conditions which are necessary and sufficient for the problem (10) and (11) to have a unique solution $\phi \equiv 0$ are $Q\phi(0) = 0$, $M\phi(0) = 0$ and $\operatorname{rank}[\mathbf{B_0}R, \mathbf{B_1}] = 2$. Here the projection matrices Q, M and R are as follows:

$$Q = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M = \mathbf{0}, \qquad R = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

It is easily verified that the additional boundary conditions $Q\phi(0) = 0$ and $M\phi(0) = 0$ required for the correct boundary conditions are

$$y_1(0) - y_2(0) = 0, \qquad y_4(0) = 0. \tag{15}$$

On the other hand, according to (12), we can easily obtain

$$\mathbf{B_0} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{B_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{16}$$

which verifies $\text{rank}[\mathbf{B_0}R, \mathbf{B_1}] = 2$.

As an aside, since the problem results from the actual flow model in a cylindrical pipe, the flow variables at the center of the pipe (i.e. $\eta = 0$) should be smooth from the physical point of view (even if the BVP (2) and (3) has the singularity at $\eta = 0$ due to the cylindrical symmetry [9]). Thus, we assume that the solutions of the BVP (2) and (3) are well-behaved or smooth at the origin (i.e. $\eta = 0$). Moreover, the condition $S\mathbf{y}(0) = 0$ is necessary for $\mathbf{y}$ to have a bounded limit for $\eta \to 0$. In fact, it is not difficult to calculate

$$S\mathbf{y}(0) = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}_{\eta=0} = \begin{pmatrix} y_2 - y_1 \\ 0 \\ y_4 \\ -y_4 \end{pmatrix}_{\eta=0} = \begin{pmatrix} f' - \frac{1}{\eta} f \\ 0 \\ \eta f''' \\ -\eta f''' \end{pmatrix}_{\eta=0}. \tag{17}$$

Since we look only for smooth solutions as mentioned earlier, we may have $\eta f'''(\eta) = 0$ at $\eta = 0$ (i.e. $y_4(0) = 0$) and additionally from $f(0) = 0$ and L'Hopital's rule we have $f'(0) = \lim_{\eta \to 0} \frac{f(\eta)}{\eta}$. So we do have $S\mathbf{y}(0) = 0$ and the right hand side of the ODE system (5) is well defined at $\eta = 0$ and can be evaluated without any problem. On the other hand, the unconventional boundary condition (i.e. the last condition of (3)) is automatically satisfied due to the smoothness of the solution at $\eta = 0$. The condition

$$y_4(0) = 0 \tag{18}$$

is also automatically satisfied and arises naturally from $S\mathbf{y}(0) = 0$ or (15). Therefore, we will replace the unconventional boundary condition (i.e. the last one in (3)) by (18). $y_1(0) = y_2(0)$ from (15) is equivalent to the condition $f(0) = 0$ since $f(0) = f(\eta) - \eta f'(\eta) + \frac{\eta^2}{2} f''(\theta\eta),\ 0 < \theta < 1$. Thus a proper set of boundary conditions are as follows:

$$y_1(0) = y_2(0), \qquad y_1(1) = 1, \qquad y_2(1) = 0, \qquad y_4(0) = 0. \tag{19}$$

Next, we obtain the smoothness result for the solution of the problem (2) and (3). For the reader's convenience, we first state a theorem in [16] for the BVP (5) and (6):

Let $\mathbf{g}(\eta, \mathbf{y}(\eta)) \in C^p[T_\rho]$, where $T_\rho = \{(\eta, \mathbf{x}) | 0 \le \eta \le 1, \mathbf{x} \in S_\rho(\mathbf{y}(\eta))\}$ and $S_\rho(\mathbf{y}(\eta)) = \{\mathbf{x} | \|\mathbf{y} - \mathbf{x}\| \le \rho, \rho > 0\},\ p \ge 0$. Then

(i) $\mathbf{y} \in C^{p+1}(0, 1]$.
(ii) $\mathbf{y} \in C^{p+1}[0, 1]$ if all eigenvalues of S have nonpositive real parts.

Note that $\mathbf{g}(\eta, \mathbf{y}(\eta))$, given by (9), satisfies the condition of the theorem above, and that all eigenvalues of S in (8) have nonpositive real parts. A straightforward application of the theorem yields that the solution $\mathbf{y}$ for the problem (2) and (3) has

$$\mathbf{y} \in C^p[0, 1], \quad p \ge 0. \tag{20}$$

3. The computational technique

The main aim of this section is to present an idea on finding multiple solutions and to solve a relevant singular IVP (see (26) and (27) in Section 3.2). In order to state them more clearly, the description of the computational technique will be separated in Sections 3.1 and 3.2, respectively.

3.1. The technique for finding multiple solutions

We now explain how we find multiple solutions of the singular BVP (5), (7)–(9) and (19). From the earlier analysis the solver bvp4c in MATLAB can be naturally applied to solve the BVP (5), (7)–(9) and (19). However, in bvp4c, an initial guess needs to be provided (this is also crucial in finding multiple solutions). According to our computational experience for this singular BVP (5) and (19), the numerical solution could be very sensitive to the initial guess. The numerical continuation technique would be crucial in obtaining a good initial guess. For example, we use a discrete set of equally spaced numbers to approximate the interval of admissible values of the Reynolds number (e.g. $Re = -3, -2, -1, 0, 1, 2, 3$). Once we obtain results for a certain value of the discrete set (e.g. $Re = 1$), these results will be used as the initial guess of the solution of

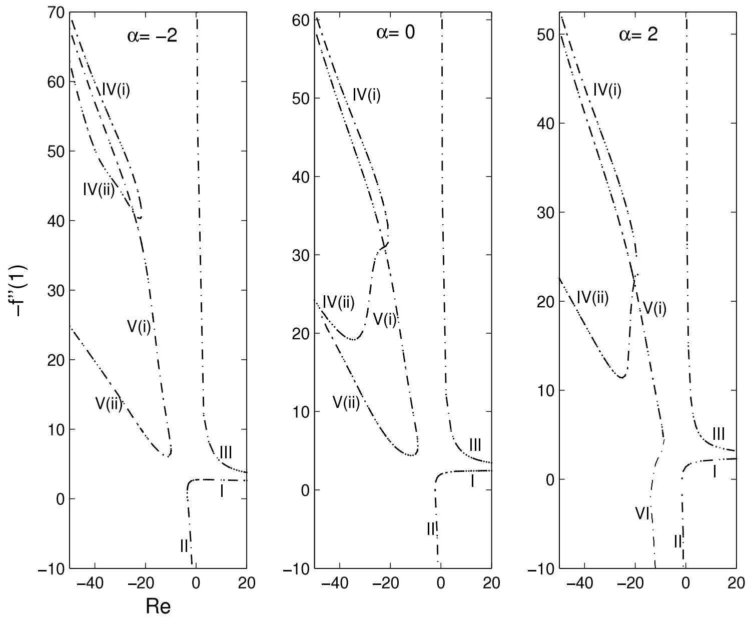

Fig. 1. Branches of $-f''(1)$ vs. Re at $\alpha = -2, 0, 2$.

the BVP at the next neighbor discrete point of the Reynolds number (e.g. $Re = 0$ or 2). Similarly, we can apply numerical continuation for the expansion ratio as well.

As we see above the numerical continuation will start from an obtained solution at certain Reynolds number. So we need to find a solution at a Reynolds number first. Also we cannot guarantee that the numerical continuation would not stop at some discrete point of the Reynolds number[1] (see also Fig. 1 for $\alpha = 0$). So we sometimes need to restart the continuation process from another solution at another Reynolds number. To achieve this, let $F = Ref$, and Eq. (1) becomes

$$8\eta F''' + 4FF'' + 8F'' + \alpha(4\eta F'' + 4F') - 4F'^2 = K, \tag{21}$$

where $K = 8Rek$. The corresponding (3) (or equivalently (19) since the solution is smooth at $\eta = 0$) becomes

$$F'(1) = 0, \qquad F(1) = Re, \qquad F(0) = 0, \qquad \lim_{\eta \to 0} \eta^{\frac{1}{2}} F''(\eta) = 0. \tag{22}$$

Then Eq. (21) will be the formulation to make use of an initial value method (to be explained in Section 3.2) for this high order ODE problem. We will use the following initial conditions:

$$F(0) = 0, \qquad F'(0) = A, \qquad F''(0) = B. \tag{23}$$

Then for $\eta = 0$ we have

$$K = 8B + 4\alpha A - 4A^2. \tag{24}$$

We will solve (21) and (23) for different A and B until finding a solution to satisfy the first condition of (22), i.e. $F'(1) = 0$ and then obtaining corresponding Reynolds number $Re = F(1)$ (here the scheme of updating A and B is based on one step of the Newton–Raphson method). The last condition of (22) is automatically satisfied for any given values of A and B due to the smoothness of the solution (see Remark 1 in Section 3.2). Thus, we can find a solution of (21) satisfying (22) for the corresponding Re. Similar ideas have been seen in literature [10,11] for different boundary conditions. But there they test all possible values of A and B until obtaining solutions for the entire range of Re. We only use the idea to find one or a few solutions and their corresponding Re to start the BVP method and the numerical continuation.

3.2. The singular IVP

To achieve the above idea, Eq. (21) is written into a first order system by introducing new variables:

$$\mathbf{z} = (z_1, z_2, z_3)^T, \quad \text{where } z_1 = F,\ z_2 = F',\ z_3 = F''. \tag{25}$$

[1] In fact the earlier studies for the case of $\alpha = 0$ revealed that there are no solutions for certain interval of Re. We expect that the numerical continuation will stop at the end points of the no-solution interval.

Then Eq. (21) becomes

$$\begin{pmatrix} z_1' \\ z_2' \\ z_3' \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ \dfrac{w(\eta,\mathbf{z})}{\eta} \end{pmatrix}, \tag{26}$$

where $w(\eta,\mathbf{z}) = \frac{K}{8} + \frac{1}{2}z_2^2 - \frac{1}{2}z_1z_3 - z_3 - \frac{\alpha}{2}(\eta z_3 + z_2)$. The corresponding initial conditions become

$$z_1(0) = 0, \qquad z_2(0) = A, \qquad z_3(0) = B. \tag{27}$$

Here, we denote $\mathbf{z}_0 = (0, A, B)^T$. Thus the problem (26) and (27) obtained from (21) and (22) is a singular IVP, where the singularity occurs at $\eta = 0$ (see $\frac{w(\eta,\mathbf{z})}{\eta}$ in (26)). It is necessary to examine whether the solution near the singular point is smooth for the singular IVP.

Before proceeding, we first introduce a result in [27]: For a nonlinear system of singular ODEs of the form

$$\eta\mathbf{u}'(\eta) = \mathbf{r}(\eta, \mathbf{u}(\eta)), \quad 0 < \eta \le T_1, \tag{28}$$

where $\mathbf{r}: [0,T_1]\times\wp \mapsto \mathbf{R}^n$ is a vector function, $\frac{\partial\mathbf{r}}{\partial\mathbf{u}}: [0,T_1]\times\wp \mapsto \mathbf{R}^{n\times n}$ is a matrix function, $\wp\subset\mathbf{R}^n$ is an open domain. The following theorem was obtained in [27].

Theorem 1. *Assume that* $\mathbf{r}\in C^m([0,T_1]\times\wp)$, $\frac{\partial\mathbf{r}}{\partial\mathbf{u}}\in C^m([0,T_1]\times\wp)$, *and that equation* $\mathbf{r}(0,\mathbf{v}) = \mathbf{0}$ *has a solution* $\mathbf{v}_0\in\wp$. *Then system (28) has for a sufficiently small* $T_2\in(0,T_1]$ *a unique solution* $\mathbf{u}\in C^m[0,T_2]$ *such that* $\mathbf{u}(0) = \mathbf{v}_0$, *where*

$$m \ge \max_{\lambda_j\in\delta(A_0)} Re\lambda_j, \quad A_0 = \left.\frac{\partial\mathbf{r}(0,\mathbf{u})}{\partial\mathbf{u}}\right|_{\mathbf{u}=\mathbf{v}_0}, \tag{29}$$

and $\delta(A_0)$ *is the set of eigenvalues of the matrix* A_0.

The system of first order ODEs (26) can be written in the following form

$$\eta\mathbf{z}'(\eta) = \tilde{w}(\eta,\mathbf{z}) = (\eta z_2, \eta z_3, w(\eta,\mathbf{z}))^T. \tag{30}$$

Since $\mathbf{z}_0$ is always a solution of the system $\tilde{w}(0,\mathbf{z}_0) = 0$ for any values of A and B, and $\tilde{w}(\eta,\mathbf{z})\in C^p([0,1]\times\wp)$, $\frac{\partial\tilde{w}(\eta,\mathbf{z})}{\partial\mathbf{z}}\in C^p([0,1]\times\wp)$ for a $p\ge 0$, according to the Theorem 1, the solvability of system (30) in $C^p[0,T_2]$ can be guaranteed. Note that the Jacobi matrix $A_0 = \frac{\partial\tilde{w}(0,\mathbf{z})}{\partial\mathbf{z}}|_{\mathbf{z}=\mathbf{z}_0}$ is

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\dfrac{1}{2}B & A - \dfrac{\alpha}{2} & -1 \end{pmatrix}, \tag{31}$$

and the eigenvalues of A_0 are 0 and -1, so (29) is satisfied for $p\ge 0$.

Remark 1. The fact that eigenvalues of A_0 are independent of α, A and B, indicates that the request for the smoothness of $\tilde{w}(\eta,\mathbf{z})$ is also independent of these parameters. In other words, for any given values of A and B, the resulting solution $\mathbf{z}$ near the singular point is always smooth, since the right hand side function $\tilde{w}$ is of polynomial (thus always smooth).

We can now evaluate the right hand side of (26) at the singular point $\eta = 0$. According to the result of previous analysis (see Remark 1), we have known that the solution near the singular point is smooth. Therefore, the evaluation of the solution at the singular point is not a problem, namely, if we let $\eta\to 0$ in (26), we find that

$$\begin{pmatrix} z_1'(0) \\ z_2'(0) \\ z_3'(0) \end{pmatrix} = \begin{pmatrix} z_2(0) \\ z_3(0) \\ \hat{w}(0,\mathbf{z}(0)) \end{pmatrix}, \tag{32}$$

where $\hat{w}(0,\mathbf{z}(0)) = \lim_{\eta\to 0}\frac{w(\eta,\mathbf{z})}{\eta}$. According to L'Hopital's rule and the smoothness property at $\eta = 0$, we can further write

$$\begin{aligned} \hat{w}(0,\mathbf{z}(0)) &= \lim_{\eta\to 0}\frac{w(\eta,\mathbf{z})}{\eta} \\ &= w'(0,\mathbf{z}(0)) \\ &= \frac{1}{2}\left(\frac{1}{2}z_2(0) - \alpha\right)z_3(0) \\ &= \frac{1}{2}\left(\frac{1}{2}A - \alpha\right)B. \end{aligned} \tag{33}$$

So the right hand side function of (26) is evaluated by (32) and (33) at the singular point. Finally, multiple solutions obtained by the technique in this section will be presented and interpreted in the next section.

Table 1
The comparison of the computation on LP for different methods.

α	Our numerical technique	AUTO	Terrill and Thomas [10]
−2	(−9.8872, −3.4711)	(−9.8901, −3.4634)	–
0	(−9.1147, −2.2997)	(−9.1125, −2.2990)	(−9.1, −2.3)
2	(−8.8493, −1.5764)	(−8.8033, −1.5785)	–

4. Multiple solutions

The main aim of this section is to present multiple solutions, namely, the BVP (i.e. (5), (7)–(9) and (19)) and the IVP (i.e. (26)–(27)) are solved. For the BVP, there exist many freely available software to solve it. These software's include MATLAB codes bvp4c [29] and bvp5c [30], sbvp [31], bvpsuite [32,33], and Fortran codes such as BVP-solver specified in [34], COLNEW described in [35] and COLSYS [36]. In this paper we simply use bvp4c built in MATLAB to solve the BVP. When concerning bvp4c solver, it is requested in the code to provide a guess for the solution desired. Further, a sufficiently good guess is very important for obtaining a good numerical solution. To provide a good guess, the numerical continuation technique described in Section 3.1 is used. As for the IVP, we simply use the solver ode45 built in MATLAB. In addition, unless stated otherwise, for all our computations, we use the relative error tolerance 10^{-3} and the absolute error tolerance 10^{-6}.

Next, our aim is to present multiple solutions of the BVP. Before presenting multiple solutions, the quantity $[-f''(1)]$ is introduced. It is proportional to the wall skin friction [10] and is usually used to show multiple solutions [7,10,11]. Here we present multiple solutions in the same way, that is, plotting $[-f''(1)]$ against Re. In Fig. 1, multiple solutions are presented for some given values of expansion ratio taken over a full range of cross-flow Reynolds number. Let us briefly explain how the computational technique presented in Section 3 is used to obtain multiple solutions in Fig. 1. Firstly, we start from $\alpha = 0$ since at this α multiple solutions information is mostly known from literature [10,11], where it is also shown that Sec. I contains all the well behaved solutions and no solution exists in the range $-9.1 < Re < -2.3$. So we start from a relatively large injection Reynolds number (e.g. $Re = 50$). According to our computational experience, in this case, the solution can be easily obtained by bvp4c for an initial guess (e.g. $[0, 0, 0, 0]$). Then, the numerical continuation is used to obtain the solution at the next neighbor Reynolds number (e.g. $Re = 49$) until near $Re = -2.3$. At this juncture, we begin to find Sec. II solution instead of Sec. I solution. However, we find that Sec. II solution cannot be easily obtained by bvp4c for arbitrarily chosen initial guesses. In this situation IVP method given in Section 3.2 is used to provide an initial guess of the solution for bvp4c. When a solution at a certain Reynolds number (denoted as $\tilde{Re}$) is obtained, the numerical continuation is again used to find the solution near $\tilde{Re}$ until near $Re = -2.3$. Finally, the continuous deformation of the velocity profiles (i.e. as $Re \to -2.3$ from above and below the limiting profiles are identical) is used to stop the numerical continuation. Next, the computational process above is extended to find other solutions and multiple solutions corresponding to $\alpha = -2, 2$.

As observed in Fig. 1, we note that the numerical results at $\alpha = 0$ are the same as what Terrill and Thomas [10] or Shankararaman and Liu [11] obtained. In a sense, this may illustrate the reliability of our numerical technique. On the other hand, the well-known bifurcation software package AUTO [37] can be used to further validate our computational results. In order to use AUTO, the BVP (i.e. (5), (7)–(9) and (19)) is converted into an autonomous system by defining $y_5 = \eta$, and the condition $y_5(0) = 0$ (or $y_5(1) = 1$) is added to (19). Then, the constant H is defined to measure the bifurcation, i.e.

$$H = \sqrt{\int_0^1 \sum_{i=1}^{5} y_i^2(\eta)\, d\eta}. \tag{34}$$

The bifurcation analysis is a process of seeking the fold point (or limit point (LP)). Due to the high sensitivity to the initial guess and the difficulty in changing the tangential direction for the problem with singularity, we have difficulty to obtain all limit points using AUTO, but we still managed to obtain a few, which are consistent with the results obtained by our method. The results are listed in the following figures and Table 1. In Figs. 2 and 3(a), the computed results of LP are shown (noting that a black dot represents a LP). Further, Table 1 presents the comparison of the computation on LP for different methods. Obviously, these results are found to be in very good agreement, indicating that our numerical technique is solid and effective in computing the multiple solution profile. In Fig. 4(a), we once again show multiple solutions in the range $-13.33 < Re < -8.8$ for the convenience of illustration. With AUTO, we start from near $Re = -12$, and the bifurcation results are showed in Fig. 4(b). During the calculation of the bifurcation, we not only find that the point F is a bifurcation point, but also detect that the point E is both a bifurcation point and a limit point (LP). These results illustrate the correctness of multiple solutions presented in Fig. 4(a). To be specific, the point E is a bifurcation point, which indicates the existence of multiple solutions (see the points C and D in Fig. 4(a)). On the other hand, the point E is a LP, which indicates the change of the solution near the point D. In other words, Sec. V(ii) solution will not exist near $Re = -40$, or there are only three solutions near $Re = -40$. The sharp decrease of the value of $-f''(1)$ corresponding to Sec. VI solution[2] in Fig. 4(a) agrees

[2] In Fig. 1, values of $-f''(1)$ for Sec. VI have not been completely included for the sake of clarity of presentation because they are much larger in magnitude compared to Sec. II. For example, the value of $-f''(1)$ is -106.8324 as $Re = -3$.

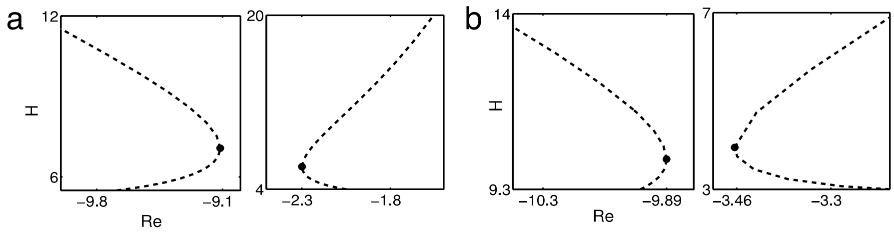

Fig. 2. Bifurcation diagrams for H at (a) $\alpha = 0$, and (b) $\alpha = -2$.

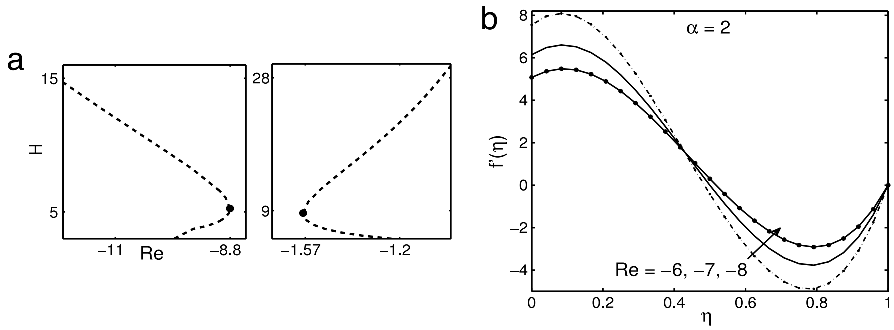

Fig. 3. Bifurcation diagrams (left) and solution Sec. VI (right) for $\alpha = 2$.

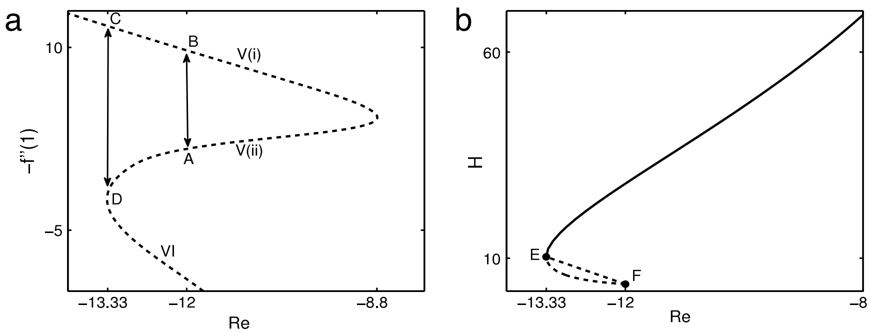

Fig. 4. Multiple solutions (left) and bifurcation diagram (right) for $\alpha = 2$.

well with the solid line in Fig. 4(b) due to $H \propto |-f''(1)|$ (see (34)). In Fig. 4, the bifurcation point F corresponds to the points A and B, and the closed curve formed by the points E and F corresponds to the curve $ABCD$. Since the Sec. VI solution is found for the first time, Fig. 3(b) shows the behavior of the axial velocity profiles $[f'(\eta)]$ (its physical meaning can be found in [10]) for different cross-flow Reynolds numbers. As for axial velocity profiles corresponding to other solutions (e.g. solution Sec. I or solution Sec. II in Fig. 1), the reader can find them in the Refs. [10,11], where the characteristic features of those solutions are summarized.

5. Asymptotic solutions as a validation of our numerical technique

In this section we construct asymptotic solutions for a certain range of Reynolds number Re and expansion ratio α. However, our goal is not to construct all possible asymptotic solutions in each category of parameters, but to obtain some of these solutions from another perspective in order to partially validate numerical solutions we obtained in the previous section.

5.1. Solution for large injection Reynolds numbers

The asymptotic solution of Eq. (1) for the large injection Reynolds number can be obtained by the Lighthill method. As indicated in [38], this is the available method which can construct a sufficiently smooth asymptotic solution for all $\eta \in [0, 1]$. Eq. (1) can be written as

$$\varepsilon(\eta f''' + f'') + \varepsilon\frac{\alpha}{2}(\eta f'' + f') + ff'' - f'^2 = \lambda, \tag{35}$$

Table 2
The numerical and asymptotic values of $-f''(1)$ for Section I.

Re	$\alpha = 0$		$\alpha = -2$		$\alpha = 2$	
	Numerical	Asymptotic	Numerical	Asymptotic	Numerical	Asymptotic
60	2.46836251	2.51726594	2.53323720	2.46493193	2.40705480	2.56959995
70	2.46878514	2.50978489	2.52479941	2.46492725	2.41547402	2.55464253
80	2.46899697	2.50417413	2.51825890	2.46492375	2.42184869	2.54342451
90	2.46908870	2.49981021	2.51305787	2.46491028	2.42683540	2.53469940
100	2.46913215	2.49631908	2.50881277	2.46491884	2.43083795	2.52771932

where

$$\varepsilon = \frac{2}{Re}, \qquad \lambda = \frac{2k}{Re}. \tag{36}$$

By introducing a new variable ξ, let

$$\eta = \xi + \varepsilon X_1(\xi) + o(\varepsilon). \tag{37}$$

$X_1(\xi)$ is unknown to be determined next. Assuming the expansion of the solution to be

$$f(\eta) = g(\xi) = \sum_{i=0}^{\infty} g_i(\xi)\varepsilon^i, \qquad \lambda = \sum_{i=0}^{\infty} \lambda_i \varepsilon^i. \tag{38}$$

Substituting Eq. (38) into Eq. (35), and equating coefficients of $\varepsilon^i (i = 0, 1, 2, \ldots)$ yields the equations

$$g_0\dddot{g}_0 - (\ddot{g}_0)^2 = \lambda_0, \tag{39}$$

$$g_0\dddot{g}_1 - 2g_0\ddot{g}_1 + \dddot{g}_0 g_1 = 2\lambda_0\dot{X}_1(\xi) + \lambda_1 - \left[\xi\dddot{g}_0 + \ddot{g}_0 + \frac{\alpha}{2}(\xi\ddot{g}_0 + \dot{g}_0)\right], \tag{40}$$

$\ldots$

where $\cdot$ denotes the derivative with respect to ξ. Assuming $\bar{\xi}$ is the root of Eq. (37) when $\eta = 1$, namely

$$1 = \bar{\xi} + \varepsilon X_1(\bar{\xi}) + \cdots. \tag{41}$$

From Eq. (41) we can obtain

$$\bar{\xi} = 1 - \varepsilon X_1(1) + \varepsilon^2[X_1(1)\dot{X}_1(1) - X_2(1)] + \cdots. \tag{42}$$

The first two conditions of Eq. (3) become

$$g_0(1) = 1, \qquad g_1(1) = \dot{g}_0(1)X_1(1), \tag{43}$$

$$\dot{g}_0(1) = 0, \qquad \dot{g}_1(1) = X_1(1)\ddot{g}_0(1). \tag{44}$$

Similarly to the above process, when $\eta = 0$, the third condition of Eq. (3) becomes

$$g_0(0) = 0, \qquad g_1(0) = X_1(0)\dot{g}_0(0). \tag{45}$$

Combining Eqs. (39) and (40) with Eqs. (43)–(45), the results are as follows:

$$\lambda_0 = -\frac{\pi^2}{4}, \qquad \lambda_1 = -\frac{\pi^2}{4} + \frac{\pi}{2} - 1, \tag{46}$$

$$X_1\left(\frac{\pi}{2}\xi\right) = \frac{\xi}{2}\sin\left(\frac{\pi}{2}\xi\right) - \left(1 + \frac{\alpha}{2}\right)\frac{\xi}{\pi}\cos\left(\frac{\pi}{2}\xi\right) + \frac{2}{\pi^2}\sin\left(\frac{\pi}{2}\xi\right) + \frac{1}{\pi}\cos\left(\frac{\pi}{2}\xi\right) + \left(\frac{1}{\pi} - \frac{2}{\pi^2} - \frac{1}{2}\right)\xi, \tag{47}$$

$$g_0(\xi) = \sin\left(\frac{\pi}{2}\xi\right), \qquad g_1(\xi) = \frac{1}{2}\cos\left(\frac{\pi}{2}\xi\right). \tag{48}$$

So $f(\eta) = g(\xi) = g_0(\xi) + \varepsilon g_1(\xi) + \cdots$ and an asymptotic approximation of $f''(1)$ can be computed. The numerical and asymptotic values of $-f''(1)$ are listed in Table 2 with expansion ratios $\alpha = 0$ and ± 2. The results agree well.

5.2. Solution for small Reynolds number and small expansion ratio

The asymptotic solution for small Reynolds number Re and small expansion ratio α can be obtained by a regular perturbation method, which is corresponding to small suction and injection for Solution Sec. I.

Let $\varepsilon = \frac{Re}{2}$ be a small perturbation parameter. Assuming the solution of Eq. (2) to be

$$f = f_0(\eta) + \varepsilon f_1(\eta) + O(\varepsilon^2), \tag{49}$$

substituting Eq. (49) into Eq. (2) and equating the coefficient of like powers of ε on both sides, we get the leading order and the first order equations as follows:

$$\eta f_0'''' + 2f_0''' + \frac{\alpha}{2}(\eta f_0''' + 2f_0'') = 0, \tag{50}$$

$$\eta f_1'''' + 2f_1''' + \frac{\alpha}{2}(\eta f_1''' + 2f_1'') + f_0 f_0''' - f_0' f_0'' = 0. \tag{51}$$

The corresponding boundary conditions of Eqs. (50) and (51) are

$$f_0(1) = 1, \qquad f_0'(1) = 0, \qquad f_0(0) = 0, \qquad \lim_{\eta\to 0} \eta^{\frac{1}{2}} f_0''(\eta) = 0, \tag{52}$$

and

$$f_1(1) = 0, \qquad f_1'(1) = 0, \qquad f_1(0) = 0, \qquad \lim_{\eta\to 0} \eta^{\frac{1}{2}} f_1''(\eta) = 0, \tag{53}$$

respectively. Secondly, because α is also small, we can use α as a secondary parameter and expand f_0, f_1 in the following forms.

$$f_i = f_{i0} + \alpha f_{i1} + O(\alpha^2), \quad i = 0, 1. \tag{54}$$

Substituting (54) into Eq. (50), we can obtain the equations of the leading and first order in α

$$\eta f_{00}'''' + 2f_{00}''' = 0, \tag{55}$$

$$f_{00}(1) = 1, \qquad f_{00}'(1) = 0, \qquad f_{00}(0) = 0, \qquad \lim_{\eta\to 0} \eta^{\frac{1}{2}} f_{00}''(\eta) = 0, \tag{56}$$

and

$$\eta f_{01}'''' + 2f_{01}''' + \frac{1}{2}(\eta f_{00}''' + 2f_{00}'') = 0, \tag{57}$$

$$f_{01}(1) = 0, \qquad f_{01}'(1) = 0, \qquad f_{01}(0) = 0, \qquad \lim_{\eta\to 0} \eta^{\frac{1}{2}} f_{01}''(\eta) = 0. \tag{58}$$

The solutions of f_{00}, f_{01} are

$$f_{00} = -\eta^2 + 2\eta, \tag{59}$$

$$f_{01} = \frac{1}{6}\eta^3 - \frac{1}{3}\eta^2 + \frac{1}{6}\eta. \tag{60}$$

In the similar process, using α as the secondary parameter, substituting (54) into Eq. (51), and collecting terms of the same order in α, we can obtain

$$\eta f_{10}'''' + 2f_{10}''' + f_{00} f_{00}''' - f_{00}' f_{00}'' = 0, \tag{61}$$

$$f_{10}(1) = 0, \qquad f_{10}'(1) = 0, \qquad f_{10}(0) = 0, \qquad \lim_{\eta\to 0} \eta^{\frac{1}{2}} f_{10}''(\eta) = 0, \tag{62}$$

and

$$\eta f_{11}'''' + 2f_{11}''' + \frac{1}{2}(\eta f_{10}''' + 2f_{10}'') + f_{00} f_{01}''' + f_{01} f_{00}''' - f_{00}' f_{01}'' - f_{01}' f_{00}'' = 0, \tag{63}$$

$$f_{11}(1) = 0, \qquad f_{11}'(1) = 0, \qquad f_{11}(0) = 0, \qquad \lim_{\eta\to 0} \eta^{\frac{1}{2}} f_{11}''(\eta) = 0, \tag{64}$$

where $f_0 = f_{00} + \alpha f_{01}$. The solutions of (61)–(64) can be obtained.

$$f_{10} = \frac{1}{18}\eta^4 - \frac{1}{3}\eta^3 + \frac{1}{2}\eta^2 - \frac{2}{9}\eta, \tag{65}$$

$$f_{11} = -\frac{1}{72}\eta^5 + \frac{17}{216}\eta^4 - \frac{2}{9}\eta^3 + \frac{19}{72}\eta^2 - \frac{23}{216}\eta. \tag{66}$$

Substituting $f_{00}, f_{01}, f_{10}, f_{11}$ into $f = f_{00} + \alpha f_{01} + \varepsilon(f_{10} + \alpha f_{11}) + O(\varepsilon^2)$, we can obtain the expression of f as the Reynolds number and expansion ratio are both small. The numerical and asymptotic values of $-f''(1)$ are compared for some values of α and Re in Table 3. The smaller the Reynolds number Re and expansion ratio α, the closer the numerical and asymptotic solutions are, indicating that our numerical computations are reliable.

Table 3
The numerical and asymptotic values of $-f''(1)$ for section I.

Re	$\alpha = 0$		$\alpha = -0.05$		$\alpha = 0.05$	
	Numerical	Asymptotic	Numerical	Asymptotic	Numerical	Asymptotic
0.592	2.0843	2.0987	2.0992	2.1133	2.0694	2.0841
0.297	2.0456	2.0495	2.0614	2.0651	2.0299	2.0339
0.041	2.0068	2.0068	2.0233	2.0234	1.9903	1.9903
−0.052	1.9912	1.9913	2.0081	2.0082	1.9744	1.9745
−0.141	1.9755	1.9765	1.9927	1.9937	1.9584	1.9593
−0.422	1.9199	1.9297	1.9383	1.9478	1.9015	1.9115

5.3. Solution for large suction Reynolds numbers

In general, there will be a boundary layer at the wall when there is large suction. The numerical results will confirm it. As the viscous terms become dominant, the perturbation solution of Eq. (1) for large suction valid outside the boundary layer would break down inside the layer and the inner solution should satisfy the conditions at the wall. In order to obtain the perturbation expansion of Eq. (1) corresponding to the large suction Reynolds number, Eq. (1) can be written as

$$\varepsilon(\eta f''' + f'') + \varepsilon\frac{\alpha}{2}(\eta f'' + f') + f'^2 - ff'' = \beta^2 + \left(\frac{\alpha}{2}\beta + \gamma\right)\varepsilon, \tag{67}$$

where

$$\varepsilon = -\frac{2}{Re}, \quad \beta^2 + \left(\frac{\alpha}{2}\beta + \gamma\right)\varepsilon = -\frac{2k}{Re}, \tag{68}$$

and

$$\beta = f'(0) = \beta_0 + \varepsilon\beta_1 + \varepsilon^2\beta_2 + \cdots, \gamma = f''(0) = \gamma_0 + \varepsilon\gamma_1 + \varepsilon^2\gamma_2 + \cdots. \tag{69}$$

From Eq. (67), the reduced problem can be obtained.

$$f_0'^2 - f_0 f_0'' = \beta_0^2, \tag{70}$$

the corresponding boundary conditions are

$$f_0(0) = 0, \qquad f_0(1) = 1, \qquad f_0'(1) = 0, \qquad \lim_{\eta\to 0} \eta^{\frac{1}{2}} f_0''(\eta) = 0. \tag{71}$$

One solution of Eq. (70) is

$$f_0 = \beta_0 \eta. \tag{72}$$

According to the condition $f(1) = 1$, one obtains

$$\beta_0 = 1, \qquad \gamma_0 = f_0''(0) = 0. \tag{73}$$

Letting

$$f = \eta + \bar{f}, \tag{74}$$

and substituting it into Eq. (67) yields

$$\varepsilon(\eta\bar{f}''' + \bar{f}'') + \varepsilon\frac{\alpha}{2}(\eta\bar{f}'' + \bar{f}') + \bar{f}'^2 + 2\bar{f}' - (\bar{f} + \eta)\bar{f}'' = 2\beta_1\varepsilon + \left(\beta_1^2 + \frac{\alpha}{2}\beta_1 + 2\beta_2 + \gamma_1\right)\varepsilon^2 + \cdots. \tag{75}$$

The asymptotic solution of Eq. (75) may be written in the form (boundary layer correction method [39–41]):

$$\bar{f} = \varepsilon(f_1(\eta) + g_1(\tau)) + \varepsilon^2(f_2(\eta) + g_2(\tau)) + \varepsilon^3(f_3(\eta) + g_3(\tau)) + \cdots, \tag{76}$$

where $\tau = \frac{1-\eta}{\varepsilon}$ is the stretching transformation near the wall and $g_i(\tau), i = 1, 2, \ldots$, are boundary layer functions (rapidly decaying when η is away from the wall). The boundary conditions satisfied by $\bar{f}(\eta)$ are

$$\bar{f}(0) = 0, \qquad \bar{f}(1) = 0, \qquad \bar{f}'(1) = -1, \qquad \lim_{\eta\to 0} \eta^{\frac{1}{2}} \bar{f}''(\eta) = 0. \tag{77}$$

Substituting (76) into Eq. (75) and equating coefficients of ε^r yields

$$\varepsilon : 2f_1' - \eta f_1'' = 2\beta_1, \tag{78}$$

$$\varepsilon^2 : 2f_2' - \eta f_2'' = \frac{\alpha}{2}\beta_1 + \gamma_1 + \beta_1^2 + 2\beta_2 - \eta f_1''' - f_1'' - \frac{\alpha}{2}\eta f_1'' - \frac{\alpha}{2}f_1' + f_1''f_1 - f_1'^2, \tag{79}$$

$$\varepsilon^3 : 2f_3' - \eta f_3'' = \frac{\alpha}{2}\beta_2 + \gamma_2 + 2\beta_1\beta_2 + 2\beta_0\beta_3 - \eta f_2''' - f_2'' - \frac{\alpha}{2}\eta f_2'' - \frac{\alpha}{2} - f_2' - 2f_1'f_2' + f_1''f_2 + f_2''f_1, \tag{80}$$

$$\begin{aligned}\varepsilon^4 : 2f_4' - \eta f_4'' \ =\ & \frac{\alpha}{2}\beta_3 + \gamma_3 + 2\beta_1\beta_3 + 2\beta_0\beta_4 + \beta_2^2 - \eta f_3''' - f_3'' - \frac{\alpha}{2}\eta f_3'' - \frac{\alpha}{2}f_3' \\ & - 2f_1'f_3' - f_2'^2 + f_1''f_3 + f_3''f_1 + f_2''f_2,\end{aligned} \tag{81}$$

$$\varepsilon^{-1} : \dddot{g}_1 + \ddot{g}_1 = 0, \tag{82}$$

$$\varepsilon^0 : \dddot{g}_2 + \ddot{g}_2 = \ddot{g}_1 + \tau\dddot{g}_1 + \frac{\alpha}{2}\ddot{g}_1 - 2\dot{g}_1 + \tau\ddot{g}_1 + \dot{g}_1^2 - \ddot{g}_1 f_1(1) - \ddot{g}_1 g_1, \tag{83}$$

$$\begin{aligned}\varepsilon : \dddot{g}_3 + \ddot{g}_3 \ =\ & \ddot{g}_2 + \tau\dddot{g}_2 + \frac{\alpha}{2}(\ddot{g}_2 - \dot{g}_1 - \tau\ddot{g}_1) - 2\dot{g}_2 + \tau\ddot{g}_2 - 2\dot{g}_1 f_1'(1) \\ & + 2\dot{g}_1\dot{g}_2 - \ddot{g}_1(f_2(1) - \tau f_1'(1)) - \ddot{g}_1 g_2 - \ddot{g}_2 f_1(1) - \ddot{g}_2 g_1,\end{aligned} \tag{84}$$

$\cdots$

where $\cdot$ and $'$ denote the derivative with respect to τ and η, respectively, and we have used $f_1(\eta) = f_1(1-\varepsilon\tau) = f_1(1) - \varepsilon\tau f_1'(1) + \frac{1}{2}\varepsilon^2\tau^2 f_1''(1) + \cdots$ and $f_2(\eta) = f_2(1-\varepsilon\tau) = f_2(1) - \varepsilon\tau f_2'(1) + \cdots$.

The boundary conditions to be satisfied by $f_i(\eta)$ and $g_i(\tau)$ at $\eta = 1$ or $\tau = 0$ are

$$f_1(0) = 0, \qquad \varepsilon g_1'(\tau)|_{\eta=1} (= -\dot{g}_1(\tau)|_{\tau=0}) = -1, \qquad f_1(\eta)|_{\eta=1} + g_1(\tau)|_{\tau=0} = 0, \tag{85}$$

$$f_i(0) = 0, \qquad f_{i-1}'(\eta)|_{\eta=1} - \dot{g}_i(\tau)|_{\tau=0} = 0, \qquad f_i(\eta)|_{\eta=1} + g_i(\tau)|_{\tau=0} = 0, \quad (i = 2, 3, 4, \ldots). \tag{86}$$

The boundary layer solution of Eq. (82) is

$$g_1(\tau) = C_1 e^{-\tau}. \tag{87}$$

From the condition $\dot{g}_1(\tau)|_{\tau=0} = 1$, one obtains $C_1 = -1$. The solution of Eq. (78) satisfying (85) is

$$f_1(\eta) = \beta_1\eta + (1-\beta_1)\eta^3. \tag{88}$$

From (88), one can obtain $\gamma_1 = f_1''(0) = 0$. The parameter β_1 is still unknown to be determined next. Substituting (88) into Eq. (79) and noticing $f_2(0) = 0$ yields

$$f_2(\eta) = (1-\beta_1)\frac{3\alpha}{2}\eta^3\ln\eta - \eta^5\left(-\frac{3\beta_1}{5} + \frac{3}{10} + \frac{3\beta_1^2}{10}\right) + \eta^3\left(\frac{A_2}{3} + \frac{\beta_1\alpha}{2} - \frac{\alpha}{2}\right) + 6\eta^2(\beta_1 - 1) + \beta_2\eta, \tag{89}$$

where A_2 is an integration constant. Since we look for the analytic solution we thus set $\beta_1 = 1$ (otherwise, $f_2'''(\eta) \to \infty$ as $\eta \to 0$). Hence,

$$f_2(\eta) = \frac{A_2}{3}\eta^3 + \beta_2\eta. \tag{90}$$

Then Eq. (83) becomes

$$\dddot{g}_2 + \ddot{g}_2 = \tau\dddot{g}_1 + \frac{\alpha}{2}\ddot{g}_1 - 2\dot{g}_1 + \tau\ddot{g}_1 + \dot{g}_1^2 - \ddot{g}_1 g_1. \tag{91}$$

Subject to the condition $f_1'(\eta)|_{\eta=1} - \dot{g}_2(\tau)|_{\tau=0} = 0$, the boundary layer solution of Eq. (91) is then

$$g_2 = -\frac{1}{2}(6 + \alpha + 4\tau + \tau\alpha)e^{-\tau}. \tag{92}$$

From the condition $f_2(\eta)|_{\eta=1} + g_2(\tau)|_{\tau=0} = 0$, one also obtains

$$f_2 = \left(3 + \frac{\alpha}{2} - \beta_2\right)\eta^3 + \beta_2\eta, \tag{93}$$

from (93),

$$\gamma_2 = f_2''(0) = 0. \tag{94}$$

Table 4
The numerical and asymptotic values of $-f''(1)$.

Re	$\alpha = 0$		$\alpha = -2$		$\alpha = 2$	
	Numerical	Asymptotic	Numerical	Asymptotic	Numerical	Asymptotic
−30.978	11.9219	14.0369	15.0322	15.2182	13.0232	12.8349
−37.952	16.3119	17.6250	18.7029	18.7665	16.3830	16.4724
−41.993	18.8162	19.6869	20.7739	20.8117	18.4633	18.5531
−51.475	24.6565	24.4957	25.5782	25.5930	23.3437	23.3923
−72.439	36.1969	35.0574	36.1180	36.1226	33.9753	33.9893
−92.48	45.1126	45.1171	46.1634	46.1665	44.0590	44.0659
−103.05	50.4133	50.4161	51.4566	51.4598	49.3652	49.3709
−122.81	60.3127	60.3152	61.3467	61.3512	59.2746	59.2781

The parameter β_2 will be determined later from the solution $f_3(\eta)$ of Eq. (80). We neglect tedious calculation and simply summarize the results below. The coefficients $\beta_2, \beta_3, \beta_4$ are:

$$\beta_2 = 3 + \frac{\alpha}{2}, \qquad \beta_3 = 18 + 4\alpha + \frac{1}{4}\alpha^2, \qquad \beta_4 = \frac{591}{4} + \frac{79}{2}\alpha + \frac{15}{4}\alpha^2 + \frac{1}{8}\alpha^3, \qquad \gamma_3 = \gamma_4 = 0, \tag{95}$$

and $f_2, f_3, f_4, g_2, g_3, g_4$ are:

$$f_2 = \left(3 + \frac{\alpha}{2}\right)\eta, \tag{96}$$

$$f_3 = \left(18 + 4\alpha + \frac{\alpha^2}{4}\right)\eta, \tag{97}$$

$$f_4 = \left(\frac{591}{4} + \frac{15\alpha^2}{4} + \frac{\alpha^3}{8} + \frac{79\alpha}{2}\right)\eta, \tag{98}$$

$$g_2 = -\frac{1}{2}(6 + \alpha + 4\tau + \tau\alpha)e^{-\tau}, \tag{99}$$

$$g_3 = -\left[\tau^2\left(\alpha + \frac{7}{2} + \frac{\alpha^2}{8}\right) + \tau\left(15 + \frac{7\alpha}{2} + \frac{\alpha^2}{4}\right) + 18 + 4\alpha + \frac{\alpha^2}{4}\right]e^{-\tau}, \tag{100}$$

$$\begin{aligned} g_4 = {} & \frac{3}{4}e^{-2\tau} - \left[\left(\frac{1}{4}\alpha^2 + \frac{1}{48}\alpha^3 + \frac{7}{4}\alpha + \frac{16}{3}\right)\tau^3 + \left(\frac{11}{8}\alpha^2 + \frac{1}{16}\alpha^3 + 12\alpha + \frac{81}{2}\right)\tau^2 \right. \\ & \left. + \left(\frac{7}{2}\alpha^2 + \frac{31}{8}\alpha^3 + \frac{71}{2}\alpha + 129\right)\tau + \left(\frac{15}{4}\alpha^2 + \frac{1}{8}\alpha^3 + \frac{79}{2}\alpha + \frac{297}{2}\right)\right]e^{-\tau}. \end{aligned} \tag{101}$$

Then the asymptotic solution for all $\eta \in [0, 1]$ is

$$f(\eta) = \eta + \varepsilon(f_1(\eta) + g_1(\tau)) + \varepsilon^2(f_2(\eta) + g_2(\tau)) + \varepsilon^3(f_3(\eta) + g_3(\tau)) + \varepsilon^4(f_4(\eta) + g_4(\tau)) + \cdots. \tag{102}$$

Then we can calculate $f''(1)$ based on the asymptotic solution above:

$$\left.\frac{d^2 f}{d\eta^2}\right|_{\eta=1} = \frac{Re}{2} + 1 + \frac{\alpha}{2} - \frac{10 + 2\alpha}{Re} + \frac{126 + 2\alpha^2 + 30\alpha}{Re^2}. \tag{103}$$

The values of (103) are compared with numerical results in Table 4, which shows that the smaller the Reynolds number Re, the closer the numerical and asymptotic values of $-f''(1)$ are.

6. Conclusions

We have investigated multiple solutions of a singular nonlinear BVP arising from the laminar flow in a porous pipe with an expanding or contracting wall. We propose a numerical technique for the singular nonlinear BVP and multiple solutions are presented for some typical values of the expansion ratio and a full range of the cross-flow Reynolds number. The numerical technique we propose in the paper has advantage over existing methods in dealing with some difficulties (e.g. the singularity and multiple parameters) in solving this singular nonlinear BVP. Based on the comparison of numerical results with AUTO and some asymptotic results, the numerical technique is robust and efficient in solving this nonlinear singular BVP for the entire range of Reynolds number Re. We believe that it can be used to some similar problems arising from fluid mechanics and other scientific fields.

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