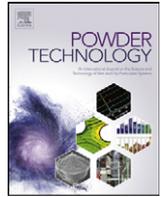




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Asymptotic solutions for laminar flow based on blood circulation through a uniformly porous channel with retractable walls and an applied transverse magnetic field

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ABSTRACT

This paper is concerned with asymptotic solutions of a nonlinear boundary value problem (BVP), which arises in a study of laminar flow in a uniformly porous channel with retractable walls and an applied transverse magnetic field. For different ranges of the control parameters (i.e. α , Re and M) arising in the BVP, four cases are considered using different singular perturbation methods. For the first case, unlike those in the existing literature, we make use of the Lighthill method and successfully construct an asymptotic solution with high-order derivatives at the center of the channel. For the second case, under large suction we consider $M^2 = O(1)$ and $M^2 = O(Re)$, respectively, which will further extend the applying range of asymptotic solutions. In other cases, asymptotic solutions with a boundary layer are successfully constructed. In addition, numerical solutions presented for each case agree well with asymptotic solutions, which illustrates that the asymptotic solutions constructed in this paper are more reliable. Finally, the influences of some parameters on flow field are discussed to develop a better understanding of the flow problem.

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1. Introduction

Blood circulating in the blood vessel has a strong effect on the human body and also serves as one of the basic substances constituting the human body. Its dynamics is closely associated with people's health. For example, as said by Srivastava [1], atherosclerosis, a leading cause of death in many countries, is one of the phenomenon in which the flow behaviour of the blood in the vessel will be influenced by the intimal thickening of stenosis artery. When severe stenosis suppresses the speed of blood, the blood supply and oxygen to the brain are reduced. Under this situation some cells in the brain start to die and then the resulting serious diseases will appear (e.g. strokes). So studies of fluid transport in the vessel can serve to better understand the functions of biological organisms (e.g. lung and cardiac).

When concerning systemic circulation in blood circulation, the blood in the left ventricle is being forced into the aorta by systole and the mitral valve between left ventricle and left atrium is closed. At this juncture the left ventricle forms a vessel with one end closed.

Meanwhile, the mass transfer of the vessel between inside and outside can be achieved by the seepage across permeable wall of the vessel [2–4]. Furthermore, some idealized mathematical models are proposed which consider the vessel to be permeable [5,6]. So studies on such flow dynamics can be meaningful in the field of bioengineering and medicine. In 1990, a mathematical model on the viscous flow of Newtonian fluid inside a permeable tube with expanding or contracting cross section was established by Goto and Uchida [7]. In their work, an expansion ratio α and a cross-flow Reynolds number Re (defined in Section 2) were introduced to measure the expansion of the pipe and the mass transfer, respectively. Later, Dauenhauer and Majdalani [8] considered the case that laminar flow in a porous channel with expanding or contracting walls and thus established a mathematical model. So far there have existed some studies on the mathematical model. To list a few, one may count Majdalani et al. [9], Asghar et al. [10] and Hang Xu et al. [11]. On the other hand, some of medical literature have also shown that certain external factors can change the hydrodynamic in blood flow. When the blood is regarded as an electrically conducting fluid, the control of blood flow can be achieved by the application of the magnetic field (Noting that the fluid is often called as Magnetohydrodynamics or MHD). Based on the experimental investigation, Karmilov [12] has revealed that the

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magnetic field exerted a most significant influence on the vascular system. Subsequently, Sambasiva [13] also studied an unsteady MHD blood flow through a porous channel with porous walls. So far, some valuable results on MHD in a pipe have been reported. For example, as said in [14], the effects of MHD on blood flow are as follows: i) to reduce the high shear stress caused by stenosis and hence to prevent the damage to the red and endothelial cells, which will help bioengineers in the design of artificial organs and the treatment of vascular diseases (e.g. [15–18]). ii) to delay the transition from laminar to turbulent flow inside the blood vessel and thus reducing high intensity shear zones, which are unfavorable to the blood and arterial wall. This may be vital to watch out for the symptoms of a carotid artery blockage (e.g. [19,20]). Motivated by above works, we have realized the importance of magnetic field appearing in a model of laminar flow in a porous pipe with expanding or contracting walls. However, very little is known so far about the result of laminar flow in a porous channel with expanding or contracting walls and an applied transverse magnetic field. Therefore, based on the work [8], a principle objective of the current study is to overcome a deficiency in their model that does not account for the presence of a magnetic field. In fact, the investigation of the steady flow of an electrically conducting viscous fluid through a semi-infinite flat plate with an applied transverse magnetic field has been initiated by Suryaprasad [21], who obtained an asymptotic solution for small Hartmann numbers (defined in Section 2). Later, Terrill and Shrestha [22,23] extended Suryaprasad's work by considering laminar flow in a porous channel with motionless walls and an applied transverse magnetic field. In their studies, based on either numerical or asymptotic approaches, some solutions were obtained for both small and large Reynolds numbers and all values of Hartmann number.

In fact, for the viscous flow in a porous channel with stationary walls, the earliest researcher can be traced back to Berman [24]. In his study, a nonlinear boundary value problem (BVP) with a cross-flow Reynolds number Re was obtained from the classical Navier-Stokes equations. For small Re , he constructed an asymptotic solution using a regular perturbation method. Subsequently, a number of further studies about the existence of multiple solutions of such a BVP followed shortly thereafter. Among these are the works of Robinson [25], Skalak and Wang [26], Shih [27], Stephen [28] and Lu [29–33]. Recently, when the walls of the channel were not motionless, Hang Xu et al. [11] obtained three solutions for large suction using homotopy analysis method (HAM). In addition, the temporal and spatial stabilities have also considerable attention in the past due to the existence of multiple solutions of the BVP, where one may count Brady [34], Durlofsky and Brady [35], Sobey and Drazin [36], Zaturka, Drazin and Banks [37].

The purpose of this paper is to extend previous investigations by presenting asymptotic solutions for laminar flow in a porous channel with expanding or contracting walls and an applied transverse magnetic field. Specifically, in Section 2, by introducing the flow geometry, governing equations with boundary conditions and a stream function, a BVP (i.e. Eqs. (13)–(14)) including three parameters (i.e. α , Re and M) is obtained. In general, when constructing a perturbation solution of the BVP, we should consider the order of magnitude among these parameters, otherwise the perturbation solution constructed is only valid for the limited scope of parameters. Therefore, Section 3 serves to present asymptotic solutions for different cases. The asymptotic and numerical solutions are compared and discussed in Section 4. Finally, Section 5 concludes the paper.

2. Mathematical formulation of the problem

We assume that the channel is of semi-infinite length with one closed end. In addition, to consider a two-dimensional flow, we assume that the distance $2a$ between the porous walls is much

smaller than the channel's width. Both sidewalls are assumed to have equal permeability $-v_w$ and to expand or contract uniformly by a time-dependent rate $\dot{a}(t)$. As shown in Fig. 1, x and y indicate the streamwise direction and the normal direction, respectively. u and v denote the velocity components along x - and y -axes. The flow velocity is zero at the closed end ($x = 0$). As a result, the motion of a viscous incompressible and electrically conducting fluid through a porous channel with an applied transverse magnetic field can be described by the following equations:

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

and

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{V} + \frac{1}{\rho} \mathbf{J} \times \mathbf{B}, \tag{2}$$

where \mathbf{J} and \mathbf{B} are given by the Maxwell equations

$$\nabla \times \mathbf{H} = 4\pi \mathbf{J}, \tag{3}$$

$$\nabla \times \mathbf{E} = 0, \tag{4}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{5}$$

and Ohm's law

$$\mathbf{J} = \sigma[\mathbf{E} + \mathbf{V} \times \mathbf{B}], \tag{6}$$

where $\mathbf{B} = \mu_m \mathbf{H}$, $\mathbf{V} = (u, v)$ and the symbols ν , σ , and μ_m represent the viscosity of the fluid, the electrical conductivity and the magnetic permeability, respectively.

For simplicity, we further assume that a constant magnetic field of strength H_0 is applied perpendicular to the walls and there is no external electric field. Meanwhile, here the induced magnetic and electric fields produced by the motion of the electrically conducting fluid are neglected. With these assumptions the magnetic term $\mathbf{J} \times \mathbf{B}$ in Eq. (2) reduces to

$$\mathbf{J} \times \mathbf{B} = -\sigma H_0^2 \mathbf{V}. \tag{7}$$

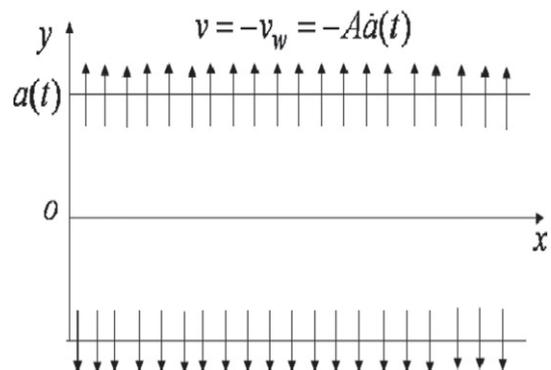


Fig. 1. Physical configuration.

When we take into account the symmetry with respect to the midsection plane, the necessary boundary conditions for the half-domain ($y \geq 0$) may be as follows:

$$u = 0, \quad v = -v_w; \quad y = a(t), \tag{8}$$

$$\frac{\partial u}{\partial y} = 0, \quad v = 0; \quad y = 0, \tag{9}$$

$$u = 0, \quad v = 0; \quad x = 0. \tag{10}$$

An appropriate stream function ϕ is introduced, denoting

$$\phi = \frac{\nu x}{h} F(y^*, t), \tag{11}$$

where $y^* = \frac{y}{a}$ is a dimensionless variable. When there is no confusion, we will delete the superscript * in the following derivation. Then the velocity components become

$$u = \frac{\nu x}{a^2} F_y, \quad v = -\frac{\nu}{a} F. \tag{12}$$

Substituting Eq. (7) and Eq. (12) into Eqs. (1)–(2) and denoting $f = \frac{F}{Re}$, with the assumptions that the wall expansion ratio α is constant and F is made dependent on y and α (see [8] for more detail on them), we can obtain the BVP of the form

$$f''' + \alpha(yf'' + 2f') + Re(ff'' - f'^2) - M^2 f' = k, \tag{13}$$

and the boundary conditions

$$f(0) = 0, \quad f''(0) = 0, \quad f(1) = 1, \quad f'(1) = 0, \tag{14}$$

where $Re = \frac{\alpha v_w}{\nu}$ is called as the cross-flow Reynolds number ($Re > 0$ for injection and $Re < 0$ for suction), $\alpha = \frac{aa'}{v}$ is called as the wall expansion ratio, $M = \mu_m H_0 a \left(\frac{\sigma}{\rho v}\right)^{\frac{1}{2}}$ is called as the Hartmann number, k is an integration constant and ' denotes differentiation with respect to y .

3. Asymptotic solutions for the BVP (i.e. Eqs. (13) and (14))

The main aim of this section is to present asymptotic solutions for these parameters (i.e. α , Re and M) with different orders of magnitude. To show them more clearly, the contents on them will be separated in Sections 3.1–3.4, respectively.

3.1. Solution for large injection Reynolds numbers

For large injection, we consider the case $\alpha = O(1)$ and $M^2 = O(1)$, and treat $\varepsilon = \frac{1}{Re}$ as the perturbation parameter. Then Eq. (13) becomes

$$\varepsilon f''' + \varepsilon \alpha (yf'' + 2f') + (ff'' - f'^2) - \varepsilon M^2 f' = \varepsilon k. \tag{15}$$

Before constructing the asymptotic solution, we first introduce a result in [38]. When $M = 0$, Majdalani and Zhou have solved

Eq. (13) asymptotically using a regular perturbation method and the variation of parameters method. The corresponding asymptotic solution is as follows:

$$f(\varsigma) = \sin \varsigma + \varepsilon \left\{ -\frac{2\alpha}{\pi} \varsigma + \left(\frac{\pi}{4} - \frac{4\alpha}{\pi}\right) \times \left[(\varsigma \cos \varsigma - \sin \varsigma) \ln \tan \frac{1}{2} \varsigma - \cos \varsigma S(\varsigma) \right] + \alpha \sin \varsigma + \left[\left(\frac{1}{2} - 8\alpha\pi^{-2}\right) S\left(\frac{1}{2}\pi\right) + 4\alpha\pi^{-2} - \frac{1}{2} \right] \varsigma \cos \varsigma \right\}, \tag{16}$$

where $\varsigma = \frac{1}{2}\pi y$ and $S(\varsigma) = \int_0^\varsigma \phi \csc \phi d\phi$. When Eq. (16) is differentiated three times, we can obtain

$$f'''(\varsigma) = \frac{1}{8}\pi^3 \left(-\cos \varsigma + \varepsilon \left\{ -\alpha \cos \varsigma + \left(\frac{1}{4}\pi - \frac{4\alpha}{\pi}\right) \times \left[-\sin \varsigma S(\varsigma) - (2 \cos \varsigma - \varsigma \sin \varsigma) \ln \tan \frac{1}{2} \varsigma - 1 \right] - \left[\left(\frac{1}{2} - 8\alpha\pi^{-2}\right) S\left(\frac{1}{2}\pi\right) + 4\alpha\pi^{-2} - \frac{1}{2} \right] (3 \cos \varsigma - \varsigma \sin \varsigma) \right\} \right), \tag{17}$$

where ' denotes differentiation with respect to ς . As observed Eq. (17), it becomes unbounded as $\varsigma \rightarrow 0$ due to the secular term $\cos \varsigma \ln \tan \frac{1}{2} \varsigma$. Besides this, when $\alpha = 0$, the series solution, given by Yuan [39], also exhibits the similar feature (i.e. the third derivative of the series solution tends to infinite at the center of the channel). However, here the unboundedness does not seem to occur in practical application. Later, the existence of a viscous shear layer, pointed out by Terrill [40], would result in the appearance of the unboundedness. To eliminate the unboundedness in f''' , Zhou and Majdalani [38] obtained a uniformly valid composite solution using matched-asymptotic expansions with logarithmic corrections. However, in the current study, we will not plan to follow their line due to the complexity of matching process. The Lighthill method (Noting that it is a method of strained coordinates, the reader can see [41] for more detail on it) is used to eliminate the secular term. The specific process is as follows: Firstly, we introduce a variable transformation on y :

$$y = \xi + \varepsilon X_1(\xi) + \varepsilon^2 X_2(\xi) + O(\varepsilon^3), \tag{18}$$

where the functions X_1, X_2 will be determined later. f and εk can be expanded as the following forms:

$$f(y) \triangleq g(\xi) = g_0(\xi) + \varepsilon g_1(\xi) + \varepsilon^2 g_2(\xi) + O(\varepsilon^3), \tag{19}$$

$$\varepsilon k = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + O(\varepsilon^3). \tag{20}$$

Substituting Eqs. (18)–(20) into Eq. (15) and equating coefficients of ε^n , one can obtain

$$\varepsilon^0 : g_0 g_0'' - g_0'^2 = \lambda_0 \tag{21}$$

$$\varepsilon : \dots g_0 + \alpha(\xi \ddot{g}_0 + 2\dot{g}_0) + g_0 \ddot{g}_1 + g_1 \ddot{g}_0 - 2\dot{g}_0 \dot{g}_1 - M^2 \dot{g}_0 = \lambda_1 + 3\dot{X}_1 \lambda_0 \dots \dots \tag{22}$$

Here \cdot denotes the derivative with respect to ξ . According to Eq. (14), the boundary conditions corresponding to $g_i (i = 0, 1, 2, \dots)$ can be induced:

i). The wall of the channel (i.e. $y = 1$)

We assume that $\tilde{\xi}$ is the root of Eq. (18) at $y = 1$, then

$$\begin{aligned} \tilde{\xi} &= 1 - \varepsilon X_1(\tilde{\xi}) - \varepsilon^2 X_2(\tilde{\xi}) + O(\varepsilon^3) \\ &= 1 - \varepsilon \left\{ X_1(1) + \dot{X}_1(1) \left[-\varepsilon X_1(\tilde{\xi}) - \varepsilon^2 X_2(\tilde{\xi}) \right] + \dots \right\} \\ &\quad - \varepsilon^2 \left\{ X_2(1) + \dot{X}_2(1) \left[-\varepsilon X_1(\tilde{\xi}) - \varepsilon^2 X_2(\tilde{\xi}) \right] + \dots \right\} \\ &\quad \dots \dots \\ &= 1 - \varepsilon X_1(1) - \varepsilon^2 \left[X_2(1) - \dot{X}_1(1) X_1(1) \right] + O(\varepsilon^3). \end{aligned} \tag{23}$$

Using Eq. (23), we can induce

$$\begin{aligned} f|_{y=1} = 1 \Rightarrow 1 = g|_{\xi=\tilde{\xi}} &= g|_{\xi=1} + \dot{g}|_{\xi=1} \\ &\times \left\{ -\varepsilon X_1(1) - \varepsilon^2 \left[X_2(1) - \dot{X}_1(1) X_1(1) \right] + \dots \right\} \\ &= g_0|_{\xi=1} + \varepsilon [g_1 - X_1 \dot{g}_0]|_{\xi=1} + O(\varepsilon^2), \end{aligned} \tag{24}$$

$$\begin{aligned} f'|_{y=1} = 0 \Rightarrow 0 = \dot{g}|_{\xi=\tilde{\xi}} &= \dot{g}|_{\xi=1} + \ddot{g}|_{\xi=1} \\ &\times \left\{ -\varepsilon X_1(1) - \varepsilon^2 \left[X_2(1) - \dot{X}_1(1) X_1(1) \right] + \dots \right\} \\ &= \dot{g}_0|_{\xi=1} + \varepsilon [\dot{g}_1 - X_1 \ddot{g}_0]|_{\xi=1} + O(\varepsilon^2). \end{aligned} \tag{25}$$

The resulting boundary conditions at $y = 1$ become

$$g_0|_{\xi=1} = 1, \quad g_1 - X_1 \dot{g}_0|_{\xi=1} = 0, \dots \tag{26}$$

$$\dot{g}_0|_{\xi=1} = 0, \quad \dot{g}_1 - X_1 \ddot{g}_0|_{\xi=1} = 0, \dots \tag{27}$$

ii). The center of the channel (i.e. $y = 0$)

We suppose that $\hat{\xi}$ is the root of Eq. (18) at $y = 0$, then

$$\begin{aligned} \hat{\xi} &= -\varepsilon X_1(\hat{\xi}) - \varepsilon^2 X_2(\hat{\xi}) + O(\varepsilon^3) = -\varepsilon X_1(0) \\ &\quad - \varepsilon^2 \left[X_2(0) - \dot{X}_1(0) X_1(0) \right] + O(\varepsilon^3), \end{aligned} \tag{28}$$

$$\begin{aligned} f|_{y=0} = 0 \Rightarrow 0 = g|_{\xi=\hat{\xi}} &= g|_{\xi=0} + \dot{g}|_{\xi=0} \\ &\times \left\{ -\varepsilon X_1(0) - \varepsilon^2 \left[X_2(0) - \dot{X}_1(0) X_1(0) \right] + \dots \right\} \\ &= g_0|_{\xi=0} + \varepsilon [g_1 - X_1 \dot{g}_0]|_{\xi=0} + O(\varepsilon^2), \end{aligned} \tag{29}$$

$$\begin{aligned} f'|_{y=0} = 0 \Rightarrow 0 = \dot{g} + \dot{g} \left[\frac{-\varepsilon \ddot{X}_1 - \varepsilon^2 \ddot{X}_2}{1 + \varepsilon \dot{X}_1 + \varepsilon^2 \dot{X}_2} \right] \Big|_{\xi=\hat{\xi}} \\ = \left\{ \dot{g}_0|_{\xi=0} + \dots g_1|_{\xi=0} \left[-\varepsilon X_1(0) - \varepsilon^2 \left(X_2(0) - \dot{X}_1(0) X_1(0) \right) + \dots \right] \right\} \\ + \left\{ \dot{g}_1|_{\xi=0} \left[-\varepsilon X_1(0) - \varepsilon^2 \left(X_2(0) - \dot{X}_1(0) X_1(0) \right) + \dots \right] \right\} \\ \cdot \left[\frac{-\varepsilon \ddot{X}_1 - \varepsilon^2 \ddot{X}_2}{1 + \varepsilon \dot{X}_1 + \varepsilon^2 \dot{X}_2} \right] \Big|_{\xi=\hat{\xi}} \\ = \dot{g}_0|_{\xi=0} + \varepsilon \left[\dot{g}_1 - X_1 \ddot{g}_0 \right] \Big|_{\xi=0} + O(\varepsilon^2). \end{aligned} \tag{30}$$

The boundary conditions at $y = 0$ become

$$g_0|_{\xi=0} = 0, \quad g_1 - X_1 \dot{g}_0|_{\xi=0} = 0, \dots \tag{31}$$

$$\dot{g}_0|_{\xi=0} = 0, \quad \dot{g}_1 - X_1 \ddot{g}_0 - \ddot{X}_1 \dot{g}_0|_{\xi=0} = 0, \dots \tag{32}$$

Using Eqs. (26)–(27) and Eqs. (31)–(32), the solution for Eq. (21) can be obtained as follows:

$$g_0 = \sin \left(\frac{\pi}{2} \xi \right) \triangleq \sin \theta, \tag{33}$$

where $\theta = \frac{\pi}{2} \xi$ and $\lambda_0 = -\frac{\pi^2}{4}$. Substituting Eq. (33) into Eq. (22) yields the equation for g_1 :

$$\begin{aligned} \sin \theta g_1''(\theta) - 2 \cos \theta g_1'(\theta) - \sin \theta g_1(\theta) &= \left(\frac{\pi}{2} - \frac{4\alpha}{\pi} + \frac{2M^2}{\pi} \right) \cos \theta \\ &\quad - 2\dot{X}_1(\xi) + \frac{2\alpha}{\pi} \theta \sin \theta + \frac{4}{\pi^2} \lambda_1, \end{aligned} \tag{34}$$

where $'$ denotes the derivative with respect to θ . To eliminate the singularity that may appear in g_1 , we can set

$$\left(\frac{\pi}{2} - \frac{4\alpha}{\pi} + \frac{2M^2}{\pi} \right) \cos \theta - 2\dot{X}_1(\xi) + \frac{2\alpha}{\pi} \theta \sin \theta + \frac{4}{\pi^2} \lambda_1 = 0. \tag{35}$$

Obviously, from Eq. (35), one can obtain

$$X_1(\theta) = \left(\frac{1}{2} - \frac{2\alpha}{\pi^2} + \frac{2M^2}{\pi^2} \right) \sin \theta - \frac{2\alpha}{\pi^2} \theta \cos \theta + \frac{4\lambda_1}{\pi^3} \theta + C_1, \tag{36}$$

where C_1 is an integration constant. At this juncture, Eq. (34) becomes

$$\sin \theta g_1''(\theta) - 2 \cos \theta g_1'(\theta) - \sin \theta g_1(\theta) = 0. \tag{37}$$

According to [9], for the general homogeneous Eq. (37), its solution can be shown as follows:

$$g_1(\theta) = K_1 \cos \theta + K_2 (\sin \theta - \theta \cos \theta), \tag{38}$$

where K_1 and K_2 are integration constants. Applying boundary conditions (26)–(27) and (31)–(32), one obtains

$$K_1 = \frac{\pi}{2} C_1, \quad K_2 = 0. \tag{39}$$

When considering the non-zero solution of g_1 and setting $C_1 = 1$, we can obtain

$$g_1 = \frac{\pi}{2} \cos \theta, \quad \lambda_1 = -\frac{\pi^2}{4} + \alpha - M^2. \tag{40}$$

Table 1
Asymptotic and numerical values of $-f''(1)$ for large injection Reynolds numbers.

Re	$\alpha = -2, M = 2$		$\alpha = 2, M = 2$		$\alpha = 2, M = 0$		
	Eq. (41)	Numerical	Eq. (41)	Numerical	Eq. (41)	Numerical	Majdalani et al. [9]
100	2.5532	2.5796	2.4647	2.4639	2.4401	2.4317	2.5041
150	2.5245	2.5420	2.4655	2.4651	2.4489	2.4434	2.4919
200	2.5102	2.5232	2.4660	2.4657	2.4535	2.4493	2.4858
250	2.5016	2.5120	2.4662	2.4660	2.4562	2.4529	2.4821
300	2.4959	2.5046	2.4664	2.4663	2.4580	2.4553	2.4796

Finally, by combining Eq. (40) with Eq. (33), the resulting solution $f(y)$ becomes as follows:

$$f(y) = \sin\left(\frac{\pi}{2}\xi\right) + \varepsilon\frac{\pi}{2}\cos\left(\frac{\pi}{2}\xi\right), \tag{41}$$

$$\text{where } y = \xi + \varepsilon\left[\left(\frac{1}{2} - \frac{2\alpha}{\pi^2} + \frac{2M^2}{\pi^2}\right)\sin\left(\frac{\pi}{2}\xi\right) - \frac{\alpha}{\pi}\xi\cos\left(\frac{\pi}{2}\xi\right) + \left(\frac{1}{2} - \frac{2\alpha}{\pi^2} + \frac{2M^2}{\pi^2}\right)\xi + 1\right].$$

To verify the validity of the asymptotic solution (41), Table 1 not only presents the comparison between asymptotic and numerical solutions (see $M = 2; \alpha = \pm 2$), but also shows the comparison our results with the analytical solution obtained by Majdalani et al. [9] for $\alpha = 2, M = 0$. As observed from Table 1, the last three columns indicate that our asymptotic results are closer to numerical results in comparison to ones obtained by Majdalani et al. In a sense, this also illustrates that applying the Lighthill method is reliable for large injection. Therefore, we hope that it can be extended to the similar flow problems.

3.2. Solution for large suction Reynolds numbers

For large suction, under $M = 0$ Majdalani et al. [8,9] have pointed out the existence of thinning boundary layer. Here we not only consider the boundary layer, but also further extend to the case $M^2 = O(Re)$. For added clarity, this section is divided into two parts: the first part- $\alpha = O(1)$ and $M^2 = O(1)$ as $Re \rightarrow -\infty$, and the second part - $\alpha = O(1)$ and $M^2 = O(Re)$ as $Re \rightarrow -\infty$.

A. $\alpha = O(1)$ and $M^2 = O(1)$ as $Re \rightarrow -\infty$

As mentioned before, a viscous boundary layer is formed near the walls of the channel. To better describe it, the method of boundary layer correction (Noting that the reader can see [42–44] for more detail on this method) is used to deal with it. The specific process is as follows:

We define $\varepsilon = \frac{1}{Re}$ to be our perturbation parameter, and set $f(0) = \beta$ and $f'''(0) = \delta$, then Eq. (13) can be written as

$$\varepsilon f'''' + \varepsilon\alpha(yf'' + 2f') + (ff'' - f'^2) - \varepsilon M^2 f' = \varepsilon\delta + 2\varepsilon\alpha\beta - \beta^2 - \varepsilon M^2\beta, \tag{42}$$

$$\text{where } \varepsilon k = \varepsilon\delta + 2\varepsilon\alpha\beta - \beta^2 - \varepsilon M^2\beta.$$

We introduce a variable transformation of the form

$$1 - y = \varepsilon\tau, \tag{43}$$

and expand the function f into a composite form as follows:

$$f(y) = f_0(y) + \varepsilon f_1(y) + g_1(\xi) + \varepsilon^2(f_2(y) + g_2(\xi)) + \varepsilon^3(f_3(y) + g_3(\xi)) + O(\varepsilon^4). \tag{44}$$

Noting that $g_i(\xi), i = 1, 2, \dots$, are boundary layer functions and rapidly decay when y is away from the walls. In addition, the constants β, δ can be written as follows:

$$\left. \begin{aligned} \beta &= \beta_0 + \varepsilon\beta_1 + \varepsilon^2\beta_2 + \varepsilon^3\beta_3 + O(\varepsilon^4) \\ \delta &= \delta_0 + \varepsilon\delta_1 + \varepsilon^2\delta_2 + \varepsilon^3\delta_3 + O(\varepsilon^4) \end{aligned} \right\}. \tag{45}$$

Using Eqs. (43)–(45), the boundary conditions in Eq. (14) become

$$f(0) = 0, f''(0) = 0 \Rightarrow f_i|_{y=0} = 0, f_i''|_{y=0} = 0, i = 0, 1, 2, \dots \tag{46}$$

$$f(1) = 1 \Rightarrow f_0|_{y=1} = 1, f_i|_{y=1} + g_i|_{\tau=0} = 0, i = 1, 2, \dots \tag{47}$$

$$f'(1) = 0 \Rightarrow f_i'|_{y=1} - \dot{g}_{i+1}|_{\tau=0} = 0, i = 0, 1, 2, \dots \tag{48}$$

Here ' and $\dot{\cdot}$ denote the derivatives with respect to y and τ , respectively. Substituting Eqs. (44)–(45) into Eq. (42) and equating coefficients of ε^n , one can obtain

$$f_0 f_0'' - f_0'^2 = -\beta_0^2. \tag{49}$$

The corresponding boundary conditions become

$$f_0|_{y=0} = 0, f_0''|_{y=0} = 0, f_0|_{y=1} = 1. \tag{50}$$

As a result, we have

$$f_0 = y, \beta_0 = 1. \tag{51}$$

For g_1 , the equation becomes

$$-\dots g_1 + \dot{g}_1 = 0, \tag{52}$$

and the corresponding boundary condition becomes

$$\dot{g}_1|_{\tau=0} = f_0'|_{y=1} = 1. \tag{53}$$

According to the character of the boundary function, the expression of g_1 should be

$$g_1 = e^\tau. \tag{54}$$

Next, we have the following equation:

$$y f_1'' - 2f_1' = -2\beta_1, \tag{55}$$

and the corresponding boundary conditions are as follows:

$$f_1|_{y=0} = 0, \quad f_1''|_{y=0} = 0, \quad f_1|_{y=1} = -g_1|_{\tau=0} = -1. \quad (56)$$

From Eqs. (55)–(56), we can obtain the following results:

$$f_1 = -y, \quad \beta_1 = -1. \quad (57)$$

At this juncture, for g_2 , the boundary condition becomes

$$\dot{g}_2|_{\tau=0} = f_1'|_{y=1} = -1. \quad (58)$$

On the other hand, the corresponding equation becomes

$$-\dots g_2 + \ddot{g}_2 + \alpha e^\tau + e^\tau - \tau e^\tau = 0. \quad (59)$$

So we can obtain

$$g_2 = e^\tau \left[-\frac{1}{2}\tau^2 + (\alpha + 3)\tau - (\alpha + 4) \right]. \quad (60)$$

Following the process above, we can easily obtain the following results:

$$f_2 = (\alpha + 4)y, \quad (61)$$

$$f_3 = \left(-\alpha^2 - 11\alpha + M^2 - \frac{129}{4} \right) y, \quad (62)$$

$$g_3 = -\frac{1}{4}e^{2\tau} + e^\tau \left[\frac{1}{8}\tau^4 - \frac{1}{2}(\alpha + 3)\tau^3 + \left(\frac{1}{2}\alpha^2 + 3\alpha + 9 \right) \tau^2 + (-\alpha^2 - 10\alpha + M^2 - 28)\tau + \alpha^2 + 11\alpha - M^2 + \frac{65}{2} \right]. \quad (63)$$

Finally, the composite solution f becomes as follows:

$$f = y + \varepsilon \{-y + e^\tau\} + \varepsilon^2 \left\{ (\alpha + 4)y + e^\tau \times \left[-\frac{1}{2}\tau^2 + (\alpha + 3)\tau - (\alpha + 4) \right] + \varepsilon^3 \left\{ \left(-\alpha^2 - 11\alpha + M^2 - \frac{129}{4} \right) y - \frac{1}{4}e^{2\tau} + e^\tau \left[\frac{1}{8}\tau^4 - \frac{1}{2}(\alpha + 3)\tau^3 + \left(\frac{1}{2}\alpha^2 + 3\alpha + 9 \right) \tau^2 + (-\alpha^2 - 10\alpha + M^2 - 28)\tau + \alpha^2 + 11\alpha - M^2 + \frac{65}{2} \right] \right\} \right\}. \quad (64)$$

A comparison of $-f'(1)$ between our results and Majdalani et al. [9] is presented in Table 2. As observed, we can find that the biggest error occurs in the case $\alpha = -20$ and $M = 52$. This is because M^2 is too large to meet the assumption $M^2 = O(1)$. So, in the next section we will consider the case $M^2 = O(Re)$.

B. $\alpha = O(1)$ and $M^2 = O(Re)$ as $Re \rightarrow -\infty$

Since $M^2 = O(Re)$ and $r = -\frac{M^2}{Re}$, we can derive $r \sim O(1)$. In addition, we denote $\varepsilon = \frac{1}{Re}$, $f'(0) = \beta$, and $f''(0) = \delta$. Eq. (13) can be written as

$$\varepsilon f''' + \varepsilon \alpha (y f'' + 2f') + (f f'' - f'^2) + r f' = \varepsilon \delta + 2\varepsilon \alpha \beta - \beta^2 + r \beta, \quad (65)$$

where $\varepsilon k = \varepsilon \delta + 2\varepsilon \alpha \beta - \beta^2 + r \beta$. Following the procedure in Part A, we can induce the expressions for f_i and g_i as follows:

$$f_0 = y, \quad (66)$$

$$f_1 = -y, \quad (67)$$

$$f_2 = (\alpha + 4 - r)y, \quad (68)$$

$$f_3 = \left(-\alpha^2 - 11\alpha + 3\alpha r - 2r^2 + 15r - \frac{129}{4} \right) y, \quad (69)$$

$$g_1 = e^\tau, \quad (70)$$

$$g_2 = e^\tau \left[-\frac{1}{2}\tau^2 + (\alpha + 3 - r)\tau - (r - \alpha - 4) \right], \quad (71)$$

$$g_3 = -\frac{1}{4}e^{2\tau} + e^\tau \left[\frac{1}{8}\tau^4 + \frac{1}{2}(r - \alpha - 3)\tau^3 + \left(\frac{1}{2}\alpha^2 + 3\alpha - \alpha r + \frac{1}{2}r^2 - 4r + 9 \right) \tau^2 + (-\alpha^2 - 10\alpha + 3\alpha r - 2r^2 + 14r - 28)\tau + \alpha^2 + 11\alpha - 3\alpha r + 2r^2 - 15r + \frac{65}{2} \right]. \quad (72)$$

Table 2
Asymptotic and numerical values of $-f'(1)$ for $\alpha = O(1)$ and $M^2 = O(1)$.

Re	$\alpha = 8, M = 4$		$\alpha = -15, M = 0$		Majdalani et al. [9]	$\alpha = -20, M = 52$	
	Eq. (64)	Numerical	Eq. (64)	Numerical		Eq. (64)	Numerical
-194	184.9253	184.9164	208.1985	208.1870	208.0001	227.2139	225.4485
-236	226.9386	226.9320	250.1631	250.1559	250.0001	266.6843	265.5398
-452	442.9679	442.9625	466.0852	466.0798	466.0000	477.1007	476.8267
-561	551.9742	551.9714	575.0686	575.0661	575.0000	584.9153	584.7459
-617	607.9765	607.9729	631.0624	631.0594	631.0000	640.4692	640.3300

Table 3
Asymptotic and numerical values of $-f''(1)$ for $\alpha = O(1)$ and $M^2 = O(Re)$.

Re	$\alpha = 8, M = 23$		$\alpha = -5, M = 31$		$\alpha = -20, M = 52$	
	Eq. (73)	Numerical	Eq. (73)	Numerical	Eq. (73)	Numerical
-194	187.7140	187.7109	202.8709	202.8725	225.1348	225.4485
-236	229.2145	229.2110	244.0378	244.0368	265.3998	265.5398
-452	444.1335	444.1283	458.1349	458.1302	476.8230	476.8267
-561	552.9089	552.9061	566.7229	566.7207	584.7450	584.7459
-617	608.8248	608.8212	622.5674	622.5642	640.3315	640.3300

As a result, the composite solution of Eq. (65) becomes as follows:

$$\begin{aligned}
 f = & y + \varepsilon(-y + e^\tau) + \varepsilon^2 \left\{ (\alpha + 4 - r)y + e^\tau \left[-\frac{1}{2}\tau^2 \right. \right. \\
 & + (\alpha + 3 - r)\tau - (r - \alpha - 4) \left. \left. \right\} + \varepsilon^3 \left\{ (-\alpha^2 - 11\alpha \right. \right. \\
 & + 3\alpha r - 2r^2 + 15r - \frac{129}{4})y - \frac{1}{4}e^{2\tau} + e^\tau \left[\frac{1}{8}\tau^4 \right. \\
 & + \frac{1}{2}(r - \alpha - 3)\tau^3 + \left(\frac{1}{2}\alpha^2 + 3\alpha - \alpha r + \frac{1}{2}r^2 - 4r + 9 \right) \tau^2 \\
 & + (-\alpha^2 - 10\alpha + 3\alpha r - 2r^2 + 14r - 28)\tau \\
 & \left. \left. + \left(\alpha^2 + 11\alpha - 3\alpha r + 2r^2 - 15r + \frac{65}{2} \right) \right] \right\}. \tag{73}
 \end{aligned}$$

When comparing Eq. (73) with Eq. (64), we can find that the Hartmann number M initially appears in the term $O(\varepsilon^2)$ for this case, which indicates the increase of the magnetic field's effect on the solution. We can observe that the asymptotic solution Eq. (73) becomes more accurate for $\alpha = -20$ and $M = 52$ (see Tables 2–3).

3.3. Solution for large Hartmann numbers

In this section, we consider the case that the intensity of the magnetic field is large enough so that the magnetic field force is the main influence factor.

We treat $\varepsilon = \frac{1}{M^2}$ as the perturbation parameter and denote $\beta = f(0)$ and $\delta = f'''(0)$, then Eq. (13) can be written as

$$\varepsilon f'''' + \varepsilon \alpha (y f'' + 2f') + \varepsilon Re (f f'' - f'^2) - f' = \varepsilon \delta + 2\varepsilon \alpha \beta - \varepsilon Re \beta^2 - \beta, \tag{74}$$

where $\varepsilon k = \varepsilon \delta + 2\varepsilon \alpha \beta - \varepsilon Re \beta^2 - \beta$.

A. Inner solution

As said by Terrill et al. [22,23], when $\alpha = 0$, large Hartmann number M would result in the appearance of a boundary layer near the wall. To obtain a solution within the boundary layer, an appropriate stretching transformation is introduced as follows:

$$\xi = \frac{1 - y}{\varepsilon^a}, \tag{75}$$

$$f(y) = 1 + \varepsilon^b g(\xi), \tag{76}$$

where a and b are constants that will be determined later. Substituting Eqs. (75)–(76) into Eq. (74) yields

$$\begin{aligned}
 -\varepsilon^{1+b-3a} \dots g + \varepsilon^{1+b-2a} A \ddot{g} - \varepsilon^{1+b-a} \alpha (\xi \ddot{g} + 2\dot{g}) \\
 + \varepsilon^{1+2b-2a} Re (g \ddot{g} - \dot{g}^2) + \varepsilon^{b-a} \dot{g} = \varepsilon \delta + 2\varepsilon \alpha \beta - \varepsilon Re \beta^2 - \beta, \tag{77}
 \end{aligned}$$

where \cdot denotes the derivative with respect to ξ and $A = \alpha + Re$.

As expected, the magnetic boundary layer involves a balance between viscous and magnetic terms. Thus we set $1 + b - 3a = b - a$, which indicates $a = \frac{1}{2}$. Besides, because of the balance of the order of magnitude on both sides of Eq. (77), we have $b = a = \frac{1}{2}$. At this juncture, Eq. (77) becomes

$$\begin{aligned}
 -\dots g + \varepsilon^{\frac{1}{2}} A \ddot{g} - \varepsilon \alpha (\xi \ddot{g} + 2\dot{g}) + \varepsilon Re (g \ddot{g} - \dot{g}^2) + \dot{g} \\
 = \varepsilon \delta + 2\varepsilon \alpha \beta - \varepsilon Re \beta^2 - \beta. \tag{78}
 \end{aligned}$$

$g(\xi)$, β and δ are expanded as follows:

$$\left. \begin{aligned}
 g(\xi) &= g_0(\xi) + \varepsilon^{\frac{1}{2}} g_1(\xi) + \varepsilon g_2(\xi) + \varepsilon^{\frac{3}{2}} g_3 + O(\varepsilon^2), \\
 \beta &= \beta_0 + \varepsilon^{\frac{1}{2}} \beta_1 + \varepsilon \beta_2 + \varepsilon^{\frac{3}{2}} \beta_3 + O(\varepsilon^2), \\
 \delta &= \delta_0 + \varepsilon^{\frac{1}{2}} \delta_1 + \varepsilon \delta_2 + \varepsilon^{\frac{3}{2}} \delta_3 + O(\varepsilon^2).
 \end{aligned} \right\} \tag{79}$$

Substituting Eq. (79) into Eq. (78) and equating coefficients of ε^i , one can obtain

$$\varepsilon^0 : -\dots g_0 + \dot{g}_0 = -\beta_0, \tag{80}$$

$$\varepsilon^{\frac{1}{2}} : -\dots g_1 + \dot{g}_1 + (\alpha + Re) \ddot{g}_0 = -\beta_1, \tag{81}$$

$$\begin{aligned}
 \varepsilon : -\dots g_2 + \dot{g}_2 + (\alpha + Re) \ddot{g}_1 - \alpha (\xi \ddot{g}_0 + 2\dot{g}_0) \\
 + Re (g_0 \ddot{g}_0 - \dot{g}_0^2) = \delta_0 + 2\alpha \beta_0 - Re \beta_0^2 - \beta_2, \tag{82}
 \end{aligned}$$

$$\begin{aligned}
 \varepsilon^{\frac{3}{2}} : -\dots g_3 + \dot{g}_3 + (\alpha + Re) \ddot{g}_2 - \alpha (\xi \ddot{g}_1 + 2\dot{g}_1) \\
 + Re (g_0 \ddot{g}_1 + g_1 \ddot{g}_0 - 2\dot{g}_0 \dot{g}_1) = \delta_1 + 2\alpha \beta_1 - 2Re \beta_0 \beta_1 - \beta_3, \tag{83}
 \end{aligned}$$

.....
The boundary conditions for g_n become

$$g_n(0) = 0, \dot{g}_n(0) = 0; n = 0, 1, 2, \dots \tag{84}$$

The solution of Eq. (80), subject to the boundary conditions in Eq. (84), is

$$g_0 = -\beta_0 e^{-\xi} - \beta_0 \xi + \beta_0. \tag{85}$$

Using Eq. (85), the solution of Eq. (81) satisfying the boundary conditions in Eq. (84) is

$$g_1 = e^{-\xi} \left(-\frac{1}{2} A \beta_0 \xi - \frac{1}{2} A \beta_0 - \beta_1 \right) - \beta_1 \xi + \left(\beta_1 + \frac{1}{2} A \beta_0 \right). \tag{86}$$

Furthermore, the expressions of g_2 and g_3 become

$$g_2 = e^{-\xi} \left[\left(\frac{1}{4} \alpha \beta_0 - \frac{1}{8} A^2 \beta_0 + \frac{1}{4} Re \beta_0^2 \right) \xi^2 + \left(-\frac{1}{4} \alpha \beta_0 - \frac{1}{8} A^2 \beta_0 - \frac{1}{2} A \beta_1 + \frac{5}{4} Re \beta_0^2 \right) \xi + \frac{5}{4} Re \beta_0^2 - \frac{1}{2} A \beta_1 - \frac{1}{4} \alpha \beta_0 - \frac{1}{8} A^2 \beta_0 + \delta_0 - \beta_2 \right] + (\delta_0 - \beta_2) \xi + \left(\frac{1}{4} \alpha \beta_0 - \frac{5}{4} Re \beta_0^2 + \frac{1}{2} A \beta_1 - \delta_0 + \beta_2 + \frac{1}{8} A^2 \beta_0 \right), \quad (87)$$

$$g_3 = e^{-\xi} \left[\left(\frac{1}{8} A \alpha \beta_0 + \frac{1}{8} A Re \beta_0^2 - \frac{1}{48} A^3 \beta_0 \right) \xi^3 + \left(-\frac{1}{8} A^2 \beta_1 + \frac{1}{4} \alpha \beta_1 + \frac{1}{2} Re \beta_0 \beta_1 + \frac{5}{8} A Re \beta_0^2 - \frac{1}{8} A \alpha \beta_0 \right) \xi^2 + \left(-\frac{1}{4} \alpha \beta_1 - \frac{3}{8} A \alpha \beta_0 - \frac{1}{2} A \beta_2 + \frac{5}{2} Re \beta_0 \beta_1 - \frac{1}{8} A^2 \beta_1 + \frac{1}{2} A \delta_0 + \frac{9}{8} A Re \beta_0^2 \right) \xi + \left(-\beta_3 + \delta_1 + \frac{5}{2} Re \beta_0 \beta_1 - \frac{3}{8} A \alpha \beta_0 + \frac{1}{2} A \delta_0 - \frac{1}{4} \alpha \beta_1 - \frac{1}{8} A^2 \beta_1 + \frac{9}{8} A Re \beta_0^2 - \frac{1}{2} A \beta_2 \right) \right] + (-\beta_3 + \delta_1) \xi + \left(\beta_3 - \delta_1 + \frac{3}{8} A \alpha \beta_0 + \frac{1}{4} \alpha \beta_1 + \frac{1}{8} A^2 \beta_1 - \frac{1}{2} A \delta_0 - \frac{9}{8} A Re \beta_0^2 + \frac{1}{2} A \beta_2 - \frac{5}{2} Re \beta_0 \beta_1 \right). \quad (88)$$

Hence, the inner perturbation solution of Eq. (74) can be expressed as

$$f^{(i)} = 1 + \varepsilon^{\frac{1}{2}} g_0 + \varepsilon g_1 + \varepsilon^{\frac{3}{2}} g_2 + \varepsilon^2 g_3 + O(\varepsilon^{\frac{5}{2}}), \quad (89)$$

where the coefficients β_n and $\delta_n (n = 0, 1, \dots)$ are to be determined by the matching process in Part C.

B. Outer solution

The outer solution of Eq. (74) satisfies the inviscid equation

$$f' = \beta, \quad (90)$$

and the corresponding outer boundary conditions become

$$f(0) = 0, f''(0) = 0. \quad (91)$$

Thus the outer solution $f^{(o)}(y)$ is

$$f^{(o)}(y) = \beta y, \quad (92)$$

where $\beta = \beta_0 + \varepsilon^{\frac{1}{2}} \beta_1 + \varepsilon \beta_2 + \varepsilon^{\frac{3}{2}} \beta_3 + \varepsilon^2 \beta_4 + O(\varepsilon^{\frac{5}{2}})$.

C. Matching process

If Eq. (92) is written in terms of the inner variable ξ , then

$$f^{(o)i} = \left(\beta_0 + \varepsilon^{\frac{1}{2}} \beta_1 + \varepsilon \beta_2 + \varepsilon^{\frac{3}{2}} \beta_3 + \varepsilon^2 \beta_4 \right) - \varepsilon^{\frac{1}{2}} \left(\beta_0 + \varepsilon^{\frac{1}{2}} \beta_1 + \varepsilon \beta_2 + \varepsilon^{\frac{3}{2}} \beta_3 \right) \xi + \dots \quad (93)$$

When $\xi \rightarrow \infty$, we can obtain

$$\beta_0 = 1, \quad \beta_1 = 1, \quad \beta_2 = 1 + \frac{1}{2} A, \quad \beta_3 = \frac{1}{8} A^2 + \frac{3}{4} (\alpha - Re) + 1, \quad (94)$$

Table 4

Asymptotic and numerical values of $-f'(1)$ for large Hartmann numbers.

M	$\alpha = 5, Re = 10$		$\alpha = -5, Re = 25$		$\alpha = 25, Re = -50$	
	Eq. (96)	Numerical	Eq. (96)	Numerical	Eq. (96)	Numerical
55	49.0888	49.0885	47.3231	47.3119	68.8821	68.8782
83	76.8921	76.8924	74.8797	74.8769	96.7608	96.7590
108	101.8021	101.8024	99.6771	99.6760	121.7030	121.7019
200	193.6637	193.6639	191.3665	191.3665	213.6118	213.6114
324	317.6012	317.6013	315.2265	315.2265	337.5696	337.5695

$$\beta_4 = \frac{1}{2} A^2 + \frac{3}{8} A (\alpha - 3Re) + \frac{3}{2} \alpha - \frac{11}{4} Re + 1, \quad \delta_0 = 0, \quad \delta_1 = 0. \quad (95)$$

As a result, the complete solution of Eq. (74) satisfying the boundary conditions in Eq. (14) can be obtained as follow:

$$f = f^{(o)} + f^{(i)} - f^{(o)i} = 1 + \varepsilon^{\frac{1}{2}} \{-e^{-\xi} - \xi + 1\} + \varepsilon \left\{ e^{-\xi} \left(-\frac{1}{2} A \xi - \frac{1}{2} A - 1 \right) - \xi + \frac{1}{2} A + 1 \right\} + \varepsilon^{\frac{3}{2}} \left\{ e^{-\xi} \left[\left(\frac{1}{4} \alpha + \frac{1}{4} Re - \frac{1}{8} A^2 \right) \xi^2 + \left(-\frac{1}{4} \alpha + \frac{5}{4} Re - \frac{1}{8} A^2 - \frac{1}{2} A \right) \xi + \left(-\frac{1}{4} \alpha + \frac{5}{4} Re - \frac{1}{8} A^2 - A - 1 \right) \right] - \left(1 + \frac{1}{2} A \right) \xi + \left(\frac{1}{4} \alpha - \frac{5}{4} Re + \frac{1}{8} A^2 + A + 1 \right) \right\} + \varepsilon^2 \left\{ e^{-\xi} \left[\left(-\frac{1}{48} A^3 + \frac{1}{8} A^2 \right) \xi^3 + \left(-\frac{1}{8} A^2 + \frac{5}{8} A Re - \frac{1}{8} A \alpha + \frac{1}{4} \alpha + \frac{1}{2} Re \right) \xi^2 + \left(-\frac{3}{8} A^2 - \frac{3}{8} A \alpha + \frac{9}{8} A Re - \frac{3}{4} \alpha + 2Re \right) \xi + \left(-\frac{1}{2} A^2 - \frac{3}{8} A \alpha + \frac{9}{8} A Re - \frac{3}{2} \alpha + \frac{11}{4} Re - 1 \right) \right] - \left(\frac{1}{8} A^2 + \frac{3}{4} \alpha - \frac{3}{4} Re + 1 \right) \xi + \frac{1}{2} A^2 + \frac{3}{8} A (\alpha - 3Re) + \frac{3}{2} \alpha - \frac{11}{4} Re + 1 \right\}. \quad (96)$$

As seen in Table 4, the asymptotic results agree well with the numerical ones no matter whether the parameters (i.e. α and Re) are positive or negative.

3.4. Solution for large wall contraction ratios

In this section, we take into account the leading influence of wall contraction ratio α on the flow, and treat $\varepsilon = -\frac{1}{\alpha}$ as the perturbation parameter. Then Eq. (13) becomes

$$-\varepsilon f''' + (y f'' + 2f') - \varepsilon Re (f f'' - f'^2) + \varepsilon M^2 f' = -\varepsilon \delta + 2\beta + \varepsilon Re \beta^2 + \varepsilon M^2 \beta, \quad (97)$$

where $-\varepsilon \delta + 2\beta + \varepsilon Re \beta^2 + \varepsilon M^2 \beta = -\varepsilon k$. When $M = 0$, Majdalani et al. [8] pointed out that the effect of large contraction ratio on flow behaviour was the same as that of large suction on flow behaviour. Thus, following the procedure outlined in Section 3.2, we introduce the stretching transformation of the form

$$\xi = \frac{1-y}{\varepsilon}, \quad f(y) = 1 + \varepsilon g(\xi). \quad (98)$$

Then Eq. (97) becomes

$$\varepsilon^{-1}(\dots g + \ddot{g}) - (\xi \ddot{g} + 2\dot{g} + Re\dot{g}) - \varepsilon Re(g\dot{g} - \dot{g}^2) - \varepsilon M^2 \dot{g} = -\varepsilon \delta + 2\beta + \varepsilon Re\beta^2 + \varepsilon M^2 \beta, \tag{99}$$

here \cdot denotes differentiation with respect to ξ . Substituting Eq. (79) into Eq. (99) and equating coefficients of ε^n yields

$$\varepsilon^{-1} : \dots g_0 + \dot{g}_0 = 0, \tag{100}$$

$$\varepsilon^0 : \dots g_1 + \dot{g}_1 - (\xi + Re)\ddot{g}_0 - 2\dot{g}_0 - 2\beta_0 = 0, \tag{101}$$

$$\varepsilon : \dots g_2 + \dot{g}_2 - (\xi + Re)\dot{g}_1 - 2\dot{g}_1 - Re(g_0\dot{g}_0 - \dot{g}_0^2) - M^2 \dot{g}_0 = -\delta_0 + 2\beta_1 + Re\beta_0^2 + M^2 \beta_0, \tag{102}$$

$$\varepsilon^2 : \dots g_3 + \dot{g}_3 - (\xi + Re)\ddot{g}_1 - 2\dot{g}_2 - Re(g_0\dot{g}_1 + g_1\dot{g}_0 - 2\dot{g}_0\dot{g}_1) - M^2 \dot{g}_1 = -\delta_1 + 2\beta_2 + 2Re\beta_0\beta_1 + M^2 \beta_1, \tag{103}$$

.....

The corresponding boundary conditions become

$$g_n(0) = 0, \quad \dot{g}_n(0) = 0; \quad n = 0, 1, 2, \dots \tag{104}$$

The outer solution $f^{(o)} = \beta y$ can be expressed in terms of the inner variable ξ :

$$f^{(o)i} = (\beta_0 + \varepsilon\beta_1 + \varepsilon^2\beta_2 + \varepsilon^3\beta_3) - \varepsilon(\beta_0 + \varepsilon\beta_1 + \varepsilon^2\beta_2)\xi + \dots \tag{105}$$

By solving Eq. (100), one can obtain

$$g_0 = C_1(e^{-\xi} + \xi - 1). \tag{106}$$

We can obtain $C_1 = -1, \beta_0 = 1$, and $\beta_1 = 1$ using the matching process. Similarly, g_1, g_2 and g_3 can be derived as follows:

$$g_1 = e^{-\xi} \left(-\frac{1}{2}\xi^2 - Re\xi - C_2 \right) + (Re - C_2)\xi + C_2, \tag{107}$$

$$g_2 = e^{-\xi} \left[-\frac{1}{8}\xi^4 - \frac{1}{2}Re\xi^3 - \frac{1}{2}(Re^2 + 1)\xi^2 + (-Re^2 + 2Re + M^2)\xi - C_3 \right] - \frac{1}{2}\delta_0\xi^2 + (Re^2 - 2Re - M^2 - C_3)\xi + C_3, \tag{108}$$

$$g_3 = e^{-\xi} \left[-\frac{1}{48}\xi^6 - \frac{1}{8}Re\xi^5 + \left(-\frac{1}{4}Re^2 + \frac{1}{8}Re - \frac{1}{8} \right) \xi^4 + \left(-\frac{1}{6}Re^3 + Re + \frac{1}{2}M^2 \right) \xi^3 + \left(-\frac{1}{2}Re^3 + \frac{5}{2}Re^2 + \frac{7}{2}Re + ReM^2 + M^2 - \frac{1}{2} \right) \xi^2 + (-Re^3 + 7Re^2 + 9Re + 3ReM^2 + 2M^2)\xi + \frac{1}{4}Re - C_4 \right] - \frac{1}{4}e^{-2\xi}Re - \frac{1}{2}\delta_1\xi^2 + \left(Re^3 - 7Re^2 - \frac{37}{4}Re - 3M^2Re - 2M^2 - C_4 \right) \xi + C_4, \tag{109}$$

where $C_2 = \beta_2 = Re + 1, C_3 = \beta_3 = Re^2 - Re + 1 - M^2$ and $C_4 = \beta_4 = Re^3 - 7Re^2 - \frac{37}{4}Re - 3M^2Re - 2M^2$. Finally, the complete solution of Eq. (97), subject to the boundary conditions in Eq. (14), can be expressed as

$$f = f^{(o)} + f^{(i)} - f^{(o)i} = 1 + \varepsilon\{-e^{-\xi} - \xi + 1\} + \varepsilon^2 \left\{ e^{-\xi} \left(-\frac{1}{2}\xi^2 - Re\xi - Re - 1 \right) - \xi + Re + 1 \right\} + \varepsilon^3 \left\{ e^{-\xi} \left[-\frac{1}{8}\xi^4 - \frac{1}{2}Re\xi^3 - \frac{1}{2}(Re^2 + 1)\xi^2 + (-Re^2 + 2Re + M^2)\xi - (Re^2 - Re + 1 - M^2) \right] + (-Re - 1)\xi + (Re^2 - Re + 1 - M^2) \right\} + \varepsilon^4 \left\{ -\frac{1}{4}e^{-2\xi}Re + e^{-\xi} \left[-\frac{1}{48}\xi^6 - \frac{1}{8}Re\xi^5 + \left(-\frac{1}{4}Re^2 + \frac{1}{8}Re - \frac{1}{8} \right) \xi^4 + \left(-\frac{1}{6}Re^3 + Re + \frac{1}{2}M^2 \right) \xi^3 + \left(-\frac{1}{2}Re^3 + \frac{5}{2}Re^2 + \frac{7}{2}Re + ReM^2 + M^2 - \frac{1}{2} \right) \xi^2 + (-Re^3 + 7Re^2 + 9Re + 3ReM^2 + 2M^2)\xi + -Re^3 + 6Re^2 + \frac{21}{2}Re + 3M^2(Re + 1) - 1 \right] + (-Re^2 + Re + M^2 - 1)\xi + \left(Re^3 - 6Re^2 - \frac{41}{4}Re - 3M^2Re - 3M^2 + 1 \right) \right\}. \tag{110}$$

A comparison of $-f'(1)$ between the asymptotic and numerical solutions is shown in Table 5. When α is large, the asymptotic results have a good match with the numerical results.

4. Comparison of the numerical and asymptotic solutions and discussion

The main aim of this section is to present a comparison of the numerical and asymptotic solutions and some discussion. For the former, the BVP (i.e. Eqs. (13) and (14)) has to be solved numerically. Here we use `bvp4c` of Matlab to solve it [45]. In addition, unless stated otherwise, for all our computations with `bvp4c`, we use the default relative error tolerance 10^{-3} and the default absolute error tolerance 10^{-6} . On the other hand, we will compare asymptotic results with numerical results to verify the accuracy of asymptotic solutions constructed in Section 3. A number of possible ways on the comparison are available, e.g. by comparing $f(y), f'(y)$ proportional to the flow velocity, $f'(1)$, related to the skin-friction at the walls, etc. In [46] Terrill pointed out that the comparison of $f'(1)$ was found to be the most effective way. Here we will follow his line, and the corresponding values of $-f'(1)$ with the different ranges of the control parameters are presented in Tables 1-5.

To develop a better understanding of the flow character, in the following section we not only graphically show the axial velocity profiles $f(y)$ over different ranges of the control parameters α, Re and M , but also further give some discussion on the effects of these parameters on flow behaviour.

When $Re = 600$, Fig. 2 illustrates the behaviour of the self-similar axial velocity $f(y)$ for expansion ratio $\alpha = -2$ and 2 , respectively, over a range of dimensionless Hartmann number M . In Fig. 2a, an initial glance indicates that the effect of varying M on the axial velocity $f(y)$ is not obvious. That is because for this case, the incompressible fluid injecting from the walls of the channel is mainly to determine the flow behaviour in comparison with the magnetic field and the wall deformation. When the Reynolds number is increased, the axial velocity distribution approaches a cosine profile (see Eq. (41) in Section 3), specifically

$$f'(y) = \cos\left(\frac{\pi}{2}y\right); \quad Re \rightarrow +\infty. \tag{111}$$

Table 5
Asymptotic and numerical values of $-f'(1)$ for large contraction ratios.

α	$M = 5, Re = -25$		$M = 23, Re = 31$		$M = 10, Re = -32$	
	Eq. (110)	Numerical	Eq. (110)	Numerical	Eq. (110)	Numerical
-150	176.7329	176.7233	125.9956	125.9916	184.0278	184.0286
-220	246.8064	246.8012	194.2248	194.2285	254.0216	254.0197
-300	326.8532	326.8509	273.2889	273.2891	334.0170	334.0160
-500	526.9088	526.9069	472.3232	472.3190	534.0109	534.0093
-654	680.9294	680.9247	625.9981	625.9954	688.0085	688.0038

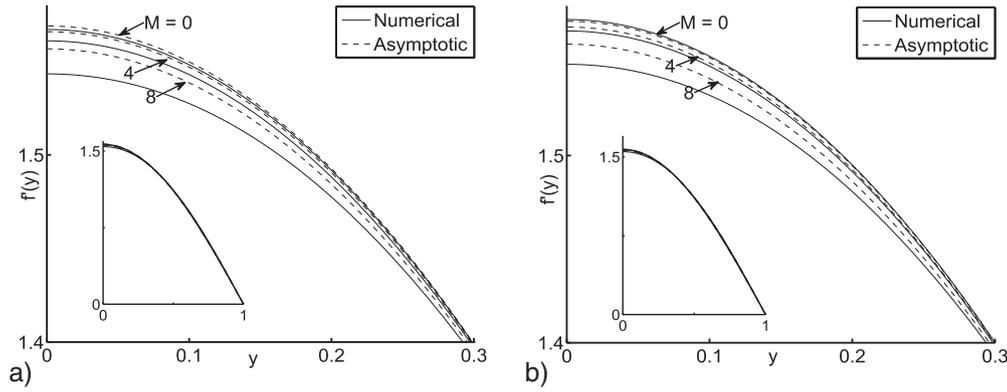


Fig. 2. Comparison between numerical and asymptotic solutions for $f(y)$ at a) $\alpha = -2$, and b) $\alpha = 2$; $Re = 600$.

Eq. (111) has often been called “Taylor’s profile” due to its relevance to several applications including paper manufacture and gas separation. With the comparison of asymptotic and numerical solutions, an error appears with Hartmann number M . Also the error increases as M increases. That is because the asymptotic solution constructed in Section 3.1 is only suitable for small Hartmann number (i.e. $M^2 = O(1)$), which can be verified by the case $M = 0$. Meanwhile, as seen in Fig. 2a, the largest error seems to occur near the center of the channel for a fixed M . When $\alpha = 2$, similar conclusions can be drawn (see Fig. 2b).

Fig. 3 presents the self-similar axial velocity $f(y)$ for large suction. In this case, since sufficiently large suction can dominate over wall expansion and magnetic field, the suction region formed near the closed head-end causes the incompressible fluid entering from the right side of the channel to move farther upstream to the walls along the wall-normal direction. Here the non-slip condition (i.e. $f(1) = 0$ in Eq. (14)) is also considered by us. As a result, the axial velocity profile becomes a spatially uniform value 1 near the center of the channel while appearing a thinning boundary layer near the walls of the channel. Moreover, from Eq. (43) in Section 3.2, we

can obtain that the boundary layer has a thickness of $O(\epsilon)$. In addition, in Fig. 3, we also see the comparison between numerical and asymptotic solutions for $f(y)$. The magnified graph indicates that a better agreement can be realized as $M \rightarrow 0$, and the error increases as M increases. However, on the whole, the accuracy of the asymptotic solution (i.e. Eq. (64) in Section 3.2) is reliable. By recalling Eqs. (64) and (73), the asymptotic solution ultimately collapses into the essentially irrotational form, namely,

$$f(y) = y + O(Re^{-1}); \quad Re \rightarrow -\infty. \tag{112}$$

Such behaviour is also consistent with Majdalani et al. [9] and Hang Xu et al. [11] in the absence of magnetic field.

To study the self-similar axial velocity sensitivity to M , the expansion ratio is held constant at $\alpha = -2$ and $\alpha = 2$ for a fixed $Re = 1$ (see Fig. 4). When the walls of the channel were motionless, Terrill and Shrestha [22] pointed out that a magnetic boundary layer existed near the walls if Re was small and M was large. Further if $Re (< 0)$ and M were both large, then the flow near the walls would

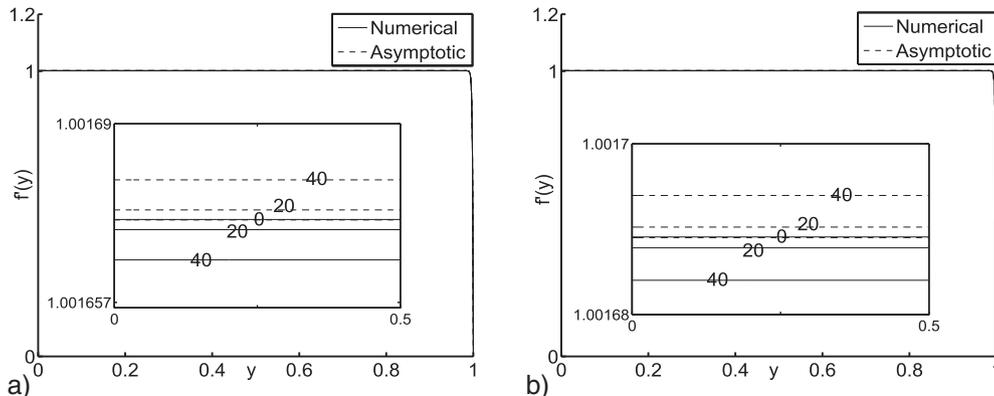


Fig. 3. Comparison between numerical and asymptotic solutions for $f(y)$ at a) $\alpha = -2$, and b) $\alpha = 2$; $Re = -600$.

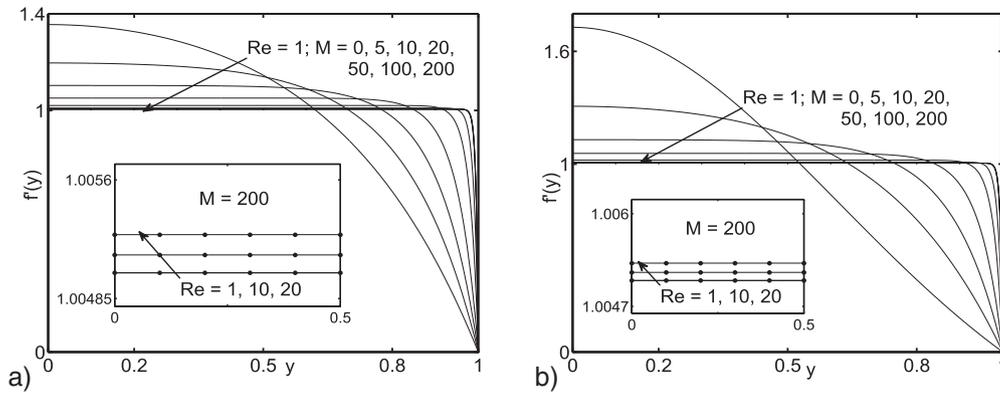


Fig. 4. Comparison between numerical and asymptotic solutions for $f(y)$ at a) $\alpha = -2$, and b) $\alpha = 2$.

consist of a combined magnetic and suction boundary layer. As predicted by them, here we do find the existence of a magnetic boundary layer (see $M = 200$). Moreover, considering the balance between viscous and magnetic terms, we also obtain that the thickness of the magnetic boundary layer is $O(\varepsilon^{\frac{1}{2}})$, which is different from the result included in Section 3.2. In addition, for $M = 200$, when Re is varied from 1 to 20, both numerical (—) and asymptotic (⋯) solutions are compared. The results are found to be in very well agreement, which indicates that the accuracy of the asymptotic solution Eq. (96) is reliable.

When $Re = 1, 15$ and $M = 0$, for varying α , the corresponding self-similar axial velocity profiles are plotted in Fig. 5. It may be interesting to note that the special case $M = 0$ and $\alpha = 0$, where the value of $f(0)$ is near 1.5 and the self-similar axial velocity profile is parabolic. Such behaviour is consistent with one in the fully developed Hagen-Poiseuille flow. As α is varied from 0 to -200 , a boundary layer is gradually formed. That is because the rapid volumetric contraction of the walls causes the incompressible fluid near the closed head-end to rapidly move towards the right side of the channel. When comparing Fig. 3 with Fig. 5, we can conclude that they have some similar phenomena, e.g. from Eq. (98), the thickness of the boundary layer is $O(\varepsilon)$, and is equal to one for large suction. when $M = 0$, to describe the similar effects of large suction and large contraction on flow behaviour, Majdalani et al. [8] have even defined an effective suction Reynolds number, that is,

$$R_s = -(Re + \alpha), \quad (113)$$

which combined the effect of the suction with the one of the wall contraction. On the other hand, for $\alpha = -200$, as M is varied

from 0 to 20, both numerical (—) and asymptotic (⋯) solutions are compared. Obviously, the accuracy of the asymptotic solution Eq. (110) is reliable, and the precision extends to 3 significant figures.

5. Conclusions

In this paper, for the BVP Eqs. (13) and (14), we have constructed asymptotic solutions for different ranges of the control parameters. Based on numerical solutions with `bvp4c`, the accuracy of these asymptotic solutions is also easily verified. On the other hand, by discussing the effects of the control parameters on the flow behaviour, some conclusions may be summarized as follows:

- i) For large injection, the axial velocity profile is still parabolic when $M^2 = O(1)$.
- ii) The Lighthill method has been successfully used to eliminate the singularity of the higher order derivatives.
- iii) Whether the situation is large suction, large Hartmann number or large contraction ratio, the boundary layer can always be found, but the corresponding thicknesses are different.

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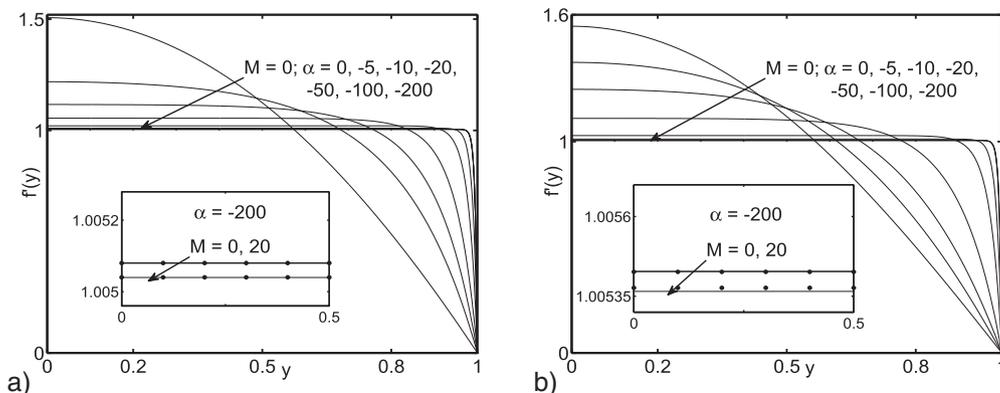


Fig. 5. Comparison between numerical and asymptotic solutions for $f(y)$ at a) $Re = 1$, and b) $Re = 15$.

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